

TEST BANK

Questions for Chapter 1

What is the negation of the propositions in 1–3?

1. Abby has more than 300 friends on facebook.
2. A messaging package for a cell phone costs less than \$20 per month.
3. $4.5 + 2.5 = 6$

In questions 4–8, determine whether the proposition is TRUE or FALSE.

4. $1 + 1 = 3$ if and only if $2 + 2 = 3$.
5. If it is raining, then it is raining.
6. If $1 < 0$, then $3 = 4$.
7. If $2 + 1 = 3$, then $2 = 3 - 1$.
8. If $1 + 1 = 2$ or $1 + 1 = 3$, then $2 + 2 = 3$ and $2 + 2 = 4$.
9. Write the truth table for the proposition $\neg(r \rightarrow \neg q) \vee (p \wedge \neg r)$.

10. (a) Find a proposition with the truth table at the right.
(b) Find a proposition using only p, q, \neg , and the connective \vee that has this truth table.

p	q	?
T	T	F
T	F	F
F	T	T
F	F	F

11. Find a proposition with three variables p, q , and r that is true when p and r are true and q is false, and false otherwise.
12. Find a proposition with three variables p, q , and r that is true when at most one of the three variables is true, and false otherwise.
13. Find a proposition with three variables p, q , and r that is never true.

14. Find a proposition using only p, q, \neg , and the connective \vee with the truth table at the right.
15. Determine whether $p \rightarrow (q \rightarrow r)$ and $p \rightarrow (q \wedge r)$ are equivalent.

p	q	?
T	T	F
T	F	T
F	T	T
F	F	F

16. Determine whether $p \rightarrow (q \rightarrow r)$ is equivalent to $(p \rightarrow q) \rightarrow r$.
17. Determine whether $(p \rightarrow q) \wedge (\neg p \rightarrow q) \equiv q$.

18. Write a proposition equivalent to $p \vee \neg q$ that uses only p, q, \neg , and the connective \wedge .
19. Write a proposition equivalent to $\neg p \wedge \neg q$ using only p, q, \neg , and the connective \vee .
20. Prove that the proposition “if it is not hot, then it is hot” is equivalent to “it is hot”.
21. Write a proposition equivalent to $p \rightarrow q$ using only p, q, \neg , and the connective \vee .
22. Write a proposition equivalent to $p \rightarrow q$ using only p, q, \neg , and the connective \wedge .
23. Prove that $p \rightarrow q$ and its converse are not logically equivalent.
24. Prove that $\neg p \rightarrow \neg q$ and its inverse are not logically equivalent.
25. Determine whether the following two propositions are logically equivalent: $p \vee (q \wedge r)$, $(p \wedge q) \vee (p \wedge r)$.
26. Determine whether the following two propositions are logically equivalent: $p \rightarrow (\neg q \wedge r)$, $\neg p \vee \neg(r \rightarrow q)$.
27. Prove that $(q \wedge (p \rightarrow \neg q)) \rightarrow \neg p$ is a tautology using propositional equivalence and the laws of logic.
28. Determine whether this proposition is a tautology: $((p \rightarrow q) \wedge \neg p) \rightarrow \neg q$.
29. Determine whether this proposition is a tautology: $((p \rightarrow \neg q) \wedge q) \rightarrow \neg p$.

In 30–36, write the statement in the form “If . . . , then”

30. x is even only if y is odd.
31. A implies B .
32. It is hot whenever it is sunny.
33. To get a good grade it is necessary that you study.
34. Studying is sufficient for passing.
35. The team wins if the quarterback can pass.
36. You need to be registered in order to check out library books.
37. Write the contrapositive, converse, and inverse of the following: If you try hard, then you will win.
38. Write the contrapositive, converse, and inverse of the following: You sleep late if it is Saturday.

In 39–41 write the negation of the statement. (Don’t write “It is not true that”)

39. It is Thursday and it is cold.
40. I will go to the play or read a book, but not both.
41. If it is rainy, then we go to the movies.
42. Explain why the negation of “Al and Bill are absent” is not “Al and Bill are present”.
43. Using c for “it is cold” and d for “it is dry”, write “It is neither cold nor dry” in symbols.
44. Using c for “it is cold” and r for “it is rainy”, write “It is rainy if it is not cold” in symbols.
45. Using c for “it is cold” and w for “it is windy”, write “To be windy it is necessary that it be cold” in symbols.
46. Using c for “it is cold”, r for “it is rainy”, and w for “it is windy”, write “It is rainy only if it is windy and cold” in symbols.

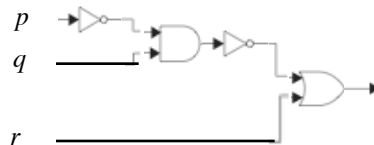
47. Translate the given statement into propositional logic using the propositions provided: On certain highways in the Washington, DC metro area you are allowed to travel on high occupancy lanes during rush hour only if there are at least three passengers in the vehicle. Express your answer in terms of r : “You are traveling during rush hour.” t : “You are riding in a car with at least three passengers.” and h : “You can travel on a high occupancy lane.”
48. A set of propositions is *consistent* if there is an assignment of truth values to each of the variables in the propositions that makes each proposition true. Is the following set of propositions consistent?
 The system is in multiuser state if and only if it is operating normally.
 If the system is operating normally, the kernel is functioning.
 The kernel is not functioning or the system is in interrupt mode.
 If the system is not in multiuser state, then it is in interrupt mode.
 The system is in interrupt mode.
49. On the island of knights and knaves you encounter two people, A and B . Person A says “ B is a knave.” Person B says “We are both knights.” Determine whether each person is a knight or a knave.
50. On the island of knights and knaves you encounter two people, A and B . Person A says “ B is a knave.” Person B says “At least one of us is a knight.” Determine whether each person is a knight or a knave.

Exercises 51–53 relate to inhabitants of an island on which there are three kinds of people: knights who always tell the truth, knaves who always lie, and spies who can either tell the truth or lie. You encounter three people, A , B , and C . You know one of the three people is a knight, one is a knave, and one is a spy. Each of the three people knows the type of person each of the other two is. For each of these situations, if possible, determine whether there is a unique solution, list all possible solutions or state that there are no solutions.

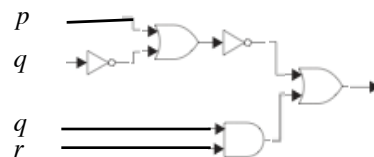
51. A says “I am not a knight,” B says “I am not a spy,” and C says “I am not a knave.”
52. A says “I am a spy,” B says “I am a spy” and C says “ B is a spy.”
53. A says “I am a knight,” B says “I am a knave,” and C says “I am not a knave.”

Find the output of the combinatorial circuits in 54–55.

54.



55.



Construct a combinatorial circuit using inverters, OR gates, and AND gates, that produces the outputs in 56–57 from input bits p , q and r .

56. $(\neg p \wedge \neg q) \vee (p \wedge \neg r)$
57. $((p \vee \neg q) \wedge r) \wedge ((\neg p \wedge \neg q) \vee r)$

Determine whether the compound propositions in 58–59 are satisfiable.

58. $(\neg p \vee \neg q) \wedge (p \rightarrow q)$
59. $(p \rightarrow q) \wedge (q \rightarrow \neg p) \wedge (p \vee q)$

In 60–62 suppose that $Q(x)$ is “ $x + 1 = 2x$ ”, where x is a real number. Find the truth value of the statement.

60. $Q(2)$.

61. $\forall x Q(x)$.

62. $\exists x Q(x)$.

In 63–70 $P(x, y)$ means “ $x + 2y = xy$ ”, where x and y are integers. Determine the truth value of the statement.

63. $P(1, -1)$.

64. $P(0, 0)$.

65. $\exists y P(3, y)$.

66. $\forall x \exists y P(x, y)$.

67. $\exists x \forall y P(x, y)$.

68. $\forall y \exists x P(x, y)$.

69. $\exists y \forall x P(x, y)$.

70. $\neg \forall x \exists y \neg P(x, y)$.

In 71–72 $P(x, y)$ means “ x and y are real numbers such that $x + 2y = 5$ ”. Determine whether the statement is true.

71. $\forall x \exists y P(x, y)$.

72. $\exists x \forall y P(x, y)$.

In 73–75 $P(m, n)$ means “ $m \leq n$ ”, where the universe of discourse for m and n is the set of nonnegative integers. What is the truth value of the statement?

73. $\forall n P(0, n)$.

74. $\exists n \forall m P(m, n)$.

75. $\forall m \exists n P(m, n)$.

In questions 76–81 suppose $P(x, y)$ is a predicate and the universe for the variables x and y is $\{1, 2, 3\}$. Suppose $P(1, 3)$, $P(2, 1)$, $P(2, 2)$, $P(2, 3)$, $P(3, 1)$, $P(3, 2)$ are true, and $P(x, y)$ is false otherwise. Determine whether the following statements are true.

76. $\forall x \exists y P(x, y)$.

77. $\exists x \forall y P(x, y)$.

78. $\neg \exists x \exists y (P(x, y) \wedge \neg P(y, x))$.

79. $\forall y \exists x (P(x, y) \rightarrow P(y, x))$.

80. $\forall x \forall y (x \neq y \rightarrow (P(x, y) \vee P(y, x)))$.

81. $\forall y \exists x (x \leq y \wedge P(x, y))$.

In 82–85 suppose the variable x represents students and y represents courses, and:

$U(y)$: y is an upper-level course $M(y)$: y is a math course $F(x)$: x is a freshman
 $B(x)$: x is a full-time student $T(x, y)$: student x is taking course y .

Write the statement using these predicates and any needed quantifiers.

82. Eric is taking MTH 281.

83. All students are freshmen.

84. Every freshman is a full-time student.

85. No math course is upper-level.

In 86–88 suppose the variable x represents students and y represents courses, and:

$U(y)$: y is an upper-level course $M(y)$: y is a math course $F(x)$: x is a freshman
 $A(x)$: x is a part-time student $T(x, y)$: student x is taking course y .

Write the statement using these predicates and any needed quantifiers.

86. Every student is taking at least one course.

87. There is a part-time student who is not taking any math course.

88. Every part-time freshman is taking some upper-level course.

In 89–91 suppose the variable x represents students and y represents courses, and:

$F(x)$: x is a freshman $A(x)$: x is a part-time student $T(x, y)$: x is taking y .

Write the statement in good English without using variables in your answers.

89. $F(\text{Mikko})$.

90. $\neg\exists y T(\text{Joe}, y)$.

91. $\exists x (A(x) \wedge \neg F(x))$.

In 92–94 suppose the variable x represents students and y represents courses, and:

$M(y)$: y is a math course $F(x)$: x is a freshman
 $B(x)$: x is a full-time student $T(x, y)$: x is taking y .

Write the statement in good English without using variables in your answers.

92. $\forall x\exists y T(x, y)$.

93. $\exists x\forall y T(x, y)$.

94. $\forall x\exists y [(B(x) \wedge F(x)) \rightarrow (M(y) \wedge T(x, y))]$.

In 95–97 suppose the variables x and y represent real numbers, and

$L(x, y)$: $x < y$ $G(x)$: $x > 0$ $P(x)$: x is a prime number.

Write the statement in good English without using any variables in your answer.

95. $L(7, 3)$.

96. $\forall x\exists y L(x, y)$.

97. $\forall x\exists y [G(x) \rightarrow (P(y) \wedge L(x, y))]$.

In 98–100 suppose the variables x and y represent real numbers, and

$L(x, y)$: $x < y$ $Q(x, y)$: $x = y$ $E(x)$: x is even $I(x)$: x is an integer.

Write the statement using these predicates and any needed quantifiers.

98. Every integer is even.

99. If $x < y$, then x is not equal to y .

100. There is no largest real number.

In 101–102 suppose the variables x and y represent real numbers, and

$$E(x) : x \text{ is even} \quad G(x) : x > 0 \quad I(x) : x \text{ is an integer.}$$

Write the statement using these predicates and any needed quantifiers.

101. Some real numbers are not positive.

102. No even integers are odd.

In 103–105 suppose the variable x represents people, and

$$F(x) : x \text{ is friendly} \quad T(x) : x \text{ is tall} \quad A(x) : x \text{ is angry.}$$

Write the statement using these predicates and any needed quantifiers.

103. Some people are not angry.

104. All tall people are friendly.

105. No friendly people are angry.

In 106–107 suppose the variable x represents people, and

$$F(x) : x \text{ is friendly} \quad T(x) : x \text{ is tall} \quad A(x) : x \text{ is angry.}$$

Write the statement using these predicates and any needed quantifiers.

106. Some tall angry people are friendly.

107. If a person is friendly, then that person is not angry.

In 108–110 suppose the variable x represents people, and

$$F(x) : x \text{ is friendly} \quad T(x) : x \text{ is tall} \quad A(x) : x \text{ is angry.}$$

Write the statement in good English. Do not use variables in your answer.

108. $A(\text{Bill})$.

109. $\neg \exists x (A(x) \wedge T(x))$.

110. $\neg \forall x (F(x) \rightarrow A(x))$.

In 111–113 suppose the variable x represents students and the variable y represents courses, and

$$A(y) : y \text{ is an advanced course} \quad S(x) : x \text{ is a sophomore} \quad F(x) : x \text{ is a freshman} \quad T(x, y) : x \text{ is taking } y.$$

Write the statement using these predicates and any needed quantifiers.

111. There is a course that every freshman is taking.

112. No freshman is a sophomore.

113. Some freshman is taking an advanced course.

In 114–115 suppose the variable x represents students and the variable y represents courses, and

$$A(y) : y \text{ is an advanced course} \quad F(x) : x \text{ is a freshman} \quad T(x, y) : x \text{ is taking } y \quad P(x, y) : x \text{ passed } y.$$

Write the statement using the above predicates and any needed quantifiers.

114. No one is taking every advanced course.

115. Every freshman passed calculus.

In 116–118 suppose the variable x represents students and the variable y represents courses, and

$$T(x, y) : x \text{ is taking } y \quad P(x, y) : x \text{ passed } y.$$

Write the statement in good English. Do not use variables in your answers.

116. $\neg P(\text{Wisteria, MAT 100})$.

117. $\exists y \forall x T(x, y)$.

118. $\forall x \exists y T(x, y)$.

In 119–123 assume that the universe for x is all people and the universe for y is the set of all movies. Write the English statement using the following predicates and any needed quantifiers:

$$S(x, y): x \text{ saw } y \quad L(x, y): x \text{ liked } y \quad A(y): y \text{ won an award} \quad C(y): y \text{ is a comedy.}$$

119. No comedy won an award.

120. Lois saw *Casablanca*, but didn't like it.

121. Some people have seen every comedy.

122. No one liked every movie he has seen.

123. Ben has never seen a movie that won an award.

In 124–126 assume that the universe for x is all people and the universe for y is the set of all movies. Write the statement in good English, using the predicates

$$S(x, y): x \text{ saw } y \quad L(x, y): x \text{ liked } y.$$

Do not use variables in your answer.

124. $\exists y \neg S(\text{Margaret}, y)$.

125. $\exists y \forall x L(x, y)$.

126. $\forall x \exists y L(x, y)$.

In 127–136 suppose the variable x represents students, y represents courses, and $T(x, y)$ means “ x is taking y ”. Match the English statement with all its equivalent symbolic statements in this list:

- | | | |
|---|--|--|
| 1. $\exists x \forall y T(x, y)$ | 2. $\exists y \forall x T(x, y)$ | 3. $\forall x \exists y T(x, y)$ |
| 4. $\neg \exists x \exists y T(x, y)$ | 5. $\exists x \forall y \neg T(x, y)$ | 6. $\forall y \exists x T(x, y)$ |
| 7. $\exists y \forall x \neg T(x, y)$ | 8. $\neg \forall x \exists y T(x, y)$ | 9. $\neg \exists y \forall x T(x, y)$ |
| 10. $\neg \forall x \exists y \neg T(x, y)$ | 11. $\neg \forall x \neg \forall y \neg T(x, y)$ | 12. $\forall x \exists y \neg T(x, y)$ |

127. Every course is being taken by at least one student.

128. Some student is taking every course.

129. No student is taking all courses.

130. There is a course that all students are taking.

131. Every student is taking at least one course.

132. There is a course that no students are taking.

133. Some students are taking no courses.

134. No course is being taken by all students.

135. Some courses are being taken by no students.

136. No student is taking any course.

In 137–147 suppose the variable x represents students, $F(x)$ means “ x is a freshman,” and $M(x)$ means “ x is a math major”. Match the statement in symbols with one of the English statements in this list:

1. Some freshmen are math majors.
2. Every math major is a freshman.
3. No math major is a freshman.

137. $\forall x (M(x) \rightarrow \neg F(x))$.
138. $\neg \exists x (M(x) \wedge \neg F(x))$.
139. $\forall x (F(x) \rightarrow \neg M(x))$.
140. $\forall x (M(x) \rightarrow F(x))$.
141. $\exists x (F(x) \wedge M(x))$.
142. $\neg \forall x (\neg F(x) \vee \neg M(x))$.
143. $\forall x (\neg (M(x) \wedge \neg F(x)))$.
144. $\forall x (\neg M(x) \vee \neg F(x))$.
145. $\neg \exists x (M(x) \wedge \neg F(x))$.
146. $\neg \exists x (M(x) \wedge F(x))$.
147. $\neg \forall x (F(x) \rightarrow \neg M(x))$.

In 148–151 let $F(A)$ be the predicate “ A is a finite set” and $S(A, B)$ be the predicate “ A is contained in B ”. Suppose the universe of discourse consists of all sets. Translate the statement into symbols.

148. Not all sets are finite.
149. Every subset of a finite set is finite.
150. No infinite set is contained in a finite set.
151. The empty set is a subset of every finite set.

In 152–156 write the negation of the statement in good English. Don’t write “It is not true that”

152. Some bananas are yellow.
153. All integers ending in the digit 7 are odd.
154. No tests are easy.
155. Roses are red and violets are blue.
156. Some skiers do not speak Swedish.
157. A student is asked to give the negation of “all bananas are ripe”.
- The student responds “all bananas are not ripe”. Explain why the English in the student’s response is ambiguous.
 - Another student says that the negation of the statement is “no bananas are ripe”. Explain why this is not correct.
 - Another student says that the negation of the statement is “some bananas are ripe”. Explain why this is not correct.
 - Give the correct negation.
158. Explain why the negation of “Some students in my class use e-mail” is not “Some students in my class do not use e-mail”.
159. What is the rule of inference used in the following:
If it snows today, the university will be closed. The university will not be closed today. Therefore, it did not snow today.

160. What is the rule of inference used in the following:

If I work all night on this homework, then I can answer all the exercises. If I answer all the exercises, I will understand the material. Therefore, if I work all night on this homework, then I will understand the material.

161. Explain why an argument of the following form is not valid:

$$\begin{array}{l} p \rightarrow q \\ \hline \neg p \\ \hline \therefore \neg q \end{array}$$

162. Determine whether the following argument is valid:

$$\begin{array}{l} p \rightarrow r \\ q \rightarrow r \\ \hline \neg(p \vee q) \\ \hline \therefore \neg r \end{array}$$

163. Determine whether the following argument is valid:

$$\begin{array}{l} p \rightarrow r \\ q \rightarrow r \\ \hline q \vee \neg r \\ \hline \therefore \neg p \end{array}$$

164. Show that the hypotheses “I left my notes in the library or I finished the rough draft of the paper” and “I did not leave my notes in the library or I revised the bibliography” imply that “I finished the rough draft of the paper or I revised the bibliography”.

165. Determine whether the following argument is valid. Name the rule of inference or the fallacy.

If n is a real number such that $n > 1$, then $n^2 > 1$. Suppose that $n^2 > 1$. Then $n > 1$.

166. Determine whether the following argument is valid. Name the rule of inference or the fallacy.

If n is a real number such that $n > 2$, then $n^2 > 4$. Suppose that $n \leq 2$. Then $n^2 \leq 4$.

167. Determine whether the following argument is valid:

She is a Math Major or a Computer Science Major.
 If she does not know discrete math, she is not a Math Major.
 If she knows discrete math, she is smart.
 She is not a Computer Science Major.
 Therefore, she is smart.

168. Determine whether the following argument is valid.

Rainy days make gardens grow.
 Gardens don't grow if it is not hot.
 It always rains on a day that is not hot.
 Therefore, if it is not hot, then it is hot.

169. Determine whether the following argument is valid.

If you are not in the tennis tournament, you will not meet Ed.
 If you aren't in the tennis tournament or if you aren't in the play, you won't meet Kelly.
 You meet Kelly or you don't meet Ed.
 It is false that you are in the tennis tournament and in the play.
 Therefore, you are in the tennis tournament.

170. Show that the premises “Every student in this class passed the first exam” and “Alvina is a student in this class” imply the conclusion “Alvina passed the first exam”.

171. Show that the premises “Jean is a student in my class” and “No student in my class is from England” imply the conclusion “Jean is not from England”.

172. Determine whether the premises “Some math majors left the campus for the weekend” and “All seniors left the campus for the weekend” imply the conclusion “Some seniors are math majors.”
173. Show that the premises “Everyone who read the textbook passed the exam”, and “Ed read the textbook” imply the conclusion “Ed passed the exam”.
174. Determine whether the premises “No juniors left campus for the weekend” and “Some math majors are not juniors” imply the conclusion “Some math majors left campus for the weekend.”
175. Show that the premise “My daughter visited Europe last week” implies the conclusion “Someone visited Europe last week”.
176. Suppose you wish to prove a theorem of the form “if p then q ”.
- If you give a direct proof, what do you assume and what do you prove?
 - If you give a proof by contraposition, what do you assume and what do you prove?
 - If you give a proof by contradiction, what do you assume and what do you prove?
177. Suppose that you had to prove a theorem of the form “if p then q ”. Explain the difference between a direct proof and a proof by contraposition.
178. Give a direct proof of the following: “If x is an odd integer and y is an even integer, then $x + y$ is odd”.
179. Give a proof by contradiction of the following: “If n is an odd integer, then n^2 is odd”.
180. Consider the following theorem: “if x and y are odd integers, then $x + y$ is even”. Give a direct proof of this theorem.
181. Consider the following theorem: “if x and y are odd integers, then $x + y$ is even”. Give a proof by contradiction of this theorem.
182. Give a proof by contradiction of the following: If x and y are even integers, then xy is even.
183. Consider the following theorem: If x is an odd integer, then $x + 2$ is odd. Give a direct proof of this theorem
184. Consider the following theorem: If x is an odd integer, then $x + 2$ is odd. Give a proof by contraposition of this theorem.
185. Consider the following theorem: If x is an odd integer, then $x + 2$ is odd. Give a proof by contradiction of this theorem.
186. Consider the following theorem: If n is an even integer, then $n + 1$ is odd. Give a direct proof of this theorem.
187. Consider the following theorem: If n is an even integer, then $n + 1$ is odd. Give a proof by contraposition of this theorem.
188. Consider the following theorem: If n is an even integer, then $n + 1$ is odd. Give a proof by contradiction of this theorem.
189. Prove that the following is true for all positive integers n : n is even if and only if $3n^2 + 8$ is even.
190. Prove the following theorem: n is even if and only if n^2 is even.
191. Prove: if m and n are even integers, then mn is a multiple of 4.
192. Prove or disprove: For all real numbers x and y , $\lfloor x - y \rfloor = \lfloor x \rfloor - \lfloor y \rfloor$.
193. Prove or disprove: For all real numbers x and y , $\lfloor x + \lfloor x \rfloor \rfloor = \lfloor 2x \rfloor$.
194. Prove or disprove: For all real numbers x and y , $\lfloor xy \rfloor = \lfloor x \rfloor \cdot \lfloor y \rfloor$.
195. Give a proof by cases that $x \leq |x|$ for all real numbers x .

196. Suppose you are allowed to give either a direct proof or a proof by contraposition of the following: if $3n + 5$ is even, then n is odd. Which type of proof would be easier to give? Explain why.
197. Prove that the following three statements about positive integers n are equivalent: (a) n is even; (b) $n^3 + 1$ is odd; (c) $n^2 - 1$ is odd.
198. Given any 40 people, prove that at least four of them were born in the same month of the year.
199. Prove that the equation $2x^2 + y^2 = 14$ has no positive integer solutions.
200. What is wrong with the following “proof” that $-3 = 3$, using backward reasoning? Assume that $-3 = 3$. Squaring both sides yields $(-3)^2 = 3^2$, or $9 = 9$. Therefore $-3 = 3$.

Answers for Chapter 1

1. Abby has fewer than 301 friends on facebook.
2. A messaging package for a cell phone costs at least \$20 per month.
3. $4.5 + 2.5 \neq 6$
4. True.
5. True.
6. True.
7. True.
8. False.

9.

p	q	r	$\neg(r \rightarrow \neg q) \vee (p \wedge \neg r)$
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	T
F	T	T	T
F	T	F	F
F	F	T	F
F	F	F	F

10. (a) $\neg p \wedge q$, (b) $\neg(p \vee \neg q)$.
11. $p \wedge \neg q \wedge r$.
12. $(p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r) \vee (\neg p \wedge \neg q \wedge \neg r)$.
13. $(p \wedge \neg p) \vee (q \wedge \neg q) \vee (r \wedge \neg r)$.
14. $\neg(\neg p \vee q) \vee \neg(p \vee \neg q)$.
15. Not equivalent. Let q be false and p and r be true.
16. Not equivalent. Let p , q , and r be false.
17. Both truth tables are identical:

p	q	$(p \rightarrow q) \wedge (\neg p \rightarrow q)$	q
T	T	T	T
T	F	F	F
F	T	T	T
F	F	F	F

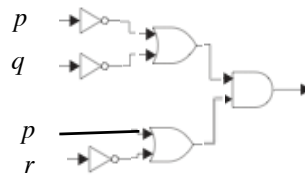
18. $\neg(\neg p \wedge q)$.
19. $\neg(p \vee q)$.
20. Both propositions are true when “it is hot” is true and both are false when “it is hot” is false.
21. $\neg p \vee q$.
22. $\neg(p \wedge \neg q)$.
23. Truth values differ when p is true and q is false.
24. Truth values differ when p is false and q is true.
25. No.
26. Yes.
27. $(q \wedge (p \rightarrow \neg q)) \rightarrow \neg p \iff (q \wedge (\neg p \vee \neg q)) \rightarrow \neg p \iff ((q \wedge \neg p) \vee (q \wedge \neg q)) \rightarrow \neg p \iff (q \wedge \neg p) \rightarrow \neg p \iff \neg(q \wedge \neg p) \vee \neg p \iff (\neg q \vee p) \vee \neg p \iff \neg q \vee (p \vee \neg p)$, which is always true.
28. No.
29. Yes.
30. If x is even, then y is odd.
31. If A , then B .
32. If it is sunny, then it is hot.
33. If you don't study, then you don't get a good grade (equivalently, if you get a good grade, then you study).
34. If you study, then you pass.
35. If the quarterback can pass, then the team wins.
36. If you are not registered, then you cannot check out library books (equivalently, if you check out library books, then you are registered).
37. Contrapositive: If you will not win, then you do not try hard. Converse: If you will win, then you try hard. Inverse: If you do not try hard, then you will not win.
38. Contrapositive: If you do not sleep late, then it is not Saturday. Converse: If you sleep late, then it is Saturday. Inverse: If it is not Saturday, then you do not sleep late.
39. It is not Thursday or it is not cold.
40. I will go to the play and read a book, or I will not go to the play and not read a book.
41. It is rainy and we do not go to the movies.
42. Both propositions can be false at the same time. For example, Al could be present and Bill absent.
43. $\neg c \wedge \neg d$.
44. $\neg c \rightarrow r$.
45. $w \rightarrow c$.
46. $r \rightarrow (w \wedge c)$.
47. $(r \wedge t) \rightarrow h$
48. Using m , n , k , and i , there are three rows of the truth table that have all five propositions true: the rows TTTT, FFFT, FFFT for m, n, k, i .
49. A is a knight, B is a knave.
50. A is a knave, B is a knight.
51. A is the spy, B is the knight, and C is the knave.
52. A is the knave, B is the spy, and C is the knight.

53. A is the knight, B is the spy, and C is the knave, or A is the knave, B is the spy, and C is the knight.

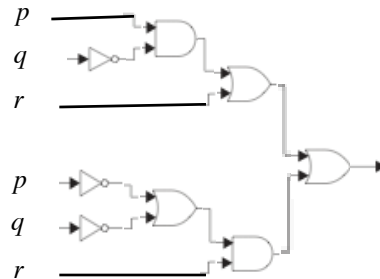
54. $\neg(\neg p \vee q) \wedge r$

55. $\neg(p \wedge \neg q) \wedge (q \vee r)$

56.



57.



58. Setting $p = \mathbf{F}$ and $q = \mathbf{T}$ makes the compound proposition true; therefore it is satisfiable.

59. Setting $q = \mathbf{T}$ and $p = \mathbf{F}$ makes the compound proposition true; therefore it is satisfiable.

60. False.

61. False.

62. True.

63. True.

64. True.

65. True.

66. False.

67. False.

68. False.

69. False.

70. False.

71. True. For every real number x we can find a real number y such that $x + 2y = 5$, namely $y = (5 - x)/2$.

72. False. If it were true for some number x_0 , then $x_0 = 5 - 2y$ for every y , which is not possible.

73. True.

74. False.

75. True.

76. True.

77. True.

78. False.

79. True.

80. False.

81. False.
82. $T(\text{Eric, MTH 281})$.
83. $\forall x F(x)$.
84. $\forall x (F(x) \rightarrow B(x))$.
85. $\forall y (M(y) \rightarrow \neg U(y))$.
86. $\forall x \exists y T(x, y)$.
87. $\exists x \forall y [A(x) \wedge (M(y) \rightarrow \neg T(x, y))]$.
88. $\forall x \exists y [(F(x) \wedge A(x)) \rightarrow (U(y) \wedge T(x, y))]$.
89. Mikko is a freshman.
90. Joe is not taking any course.
91. Some part-time students are not freshmen.
92. Every student is taking a course.
93. Some student is taking every course.
94. Every full-time freshman is taking a math course.
95. $7 < 3$.
96. There is no largest number.
97. No matter what positive number is chosen, there is a larger prime.
98. $\forall x (I(x) \rightarrow E(x))$.
99. $\forall x \forall y (L(x, y) \rightarrow \neg Q(x, y))$.
100. $\forall x \exists y L(x, y)$.
101. $\exists x \neg G(x)$.
102. $\neg \exists x (I(x) \wedge E(x) \wedge \neg E(x))$.
103. $\exists x \neg A(x)$.
104. $\forall x (T(x) \rightarrow F(x))$.
105. $\forall x (F(x) \rightarrow \neg A(x))$.
106. $\exists x (T(x) \wedge A(x) \wedge F(x))$.
107. $\forall x (F(x) \rightarrow \neg A(x))$.
108. Bill is angry.
109. No one is tall and angry.
110. Some friendly people are not angry.
111. $\exists y \forall x (F(x) \rightarrow T(x, y))$.
112. $\neg \exists x (F(x) \wedge S(x))$.
113. $\exists x \exists y (F(x) \wedge A(y) \wedge T(x, y))$.
114. $\neg \exists x \forall y (A(y) \rightarrow T(x, y))$.
115. $\forall x (F(x) \rightarrow P(x, \text{calculus}))$.
116. Wisteria did not pass MAT 100.
117. There is a course that all students are taking.
118. Every student is taking at least one course.

119. $\forall y (C(y) \rightarrow \neg A(y))$.
120. $S(\text{Lois, Casablanca}) \wedge \neg L(\text{Lois, Casablanca})$.
121. $\exists x \forall y [C(y) \rightarrow S(x, y)]$.
122. $\neg \exists x \forall y [S(x, y) \rightarrow L(x, y)]$.
123. $\neg \exists y [A(y) \wedge S(\text{Ben, } y)]$.
124. There is a movie that Margaret did not see.
125. There is a movie that everyone liked.
126. Everyone liked at least one movie.
127. 6.
128. 1, 10.
129. 12.
130. 2.
131. 3.
132. 7.
133. 5, 8, 11.
134. 9.
135. 7.
136. 4.
137. 3.
138. 2.
139. 3.
140. 2.
141. 1.
142. 1.
143. 2.
144. 3.
145. 2.
146. 3.
147. 1.
148. $\exists A \neg F(A)$.
149. $\forall A \forall B [(F(B) \wedge S(A, B)) \rightarrow F(A)]$.
150. $\neg \exists A \exists B (\neg F(A) \wedge F(B) \wedge S(A, B))$.
151. $\forall A (F(A) \rightarrow S(\emptyset, A))$.
152. No bananas are yellow.
153. Some integers ending in the digit 7 are not odd.
154. Some tests are easy.
155. Roses are not red or violets are not blue.
156. All skiers speak Swedish.

157. (a) Depending on which word is emphasized, the sentence can be interpreted as “all bananas are non-ripe fruit” (i.e., no bananas are ripe) or as “not all bananas are ripe” (i.e., some bananas are not ripe).
 (b) Both statements can be false at the same time.
 (c) Both statements can be true at the same time.
 (d) Some bananas are not ripe.
158. Both statements can be true at the same time.
159. Modus tollens.
160. Hypothetical syllogism.
161. p false and q true yield true hypotheses but a false conclusion.
162. Not valid: p false, q false, r true.
163. Not valid: p true, q true, r true.
164. Use resolution on $l \vee f$ and $\neg l \vee r$ to conclude $f \vee r$.
165. Not valid: fallacy of affirming the conclusion.
166. Not valid: fallacy of denying the hypothesis.
167. Valid.
168. Valid.
169. Not valid.
170. Universal instantiation.
171. Universal instantiation.
172. The two premises do not imply the conclusion.
173. Let $R(x)$ be the predicate “ x has read the textbook” and $P(x)$ be the predicate “ x passed the exam”. The following is the proof:
- | | |
|--|------------------------------|
| 1. $\forall x (R(x) \rightarrow P(x))$ | hypothesis |
| 2. $R(\text{Ed}) \rightarrow P(\text{Ed})$ | universal instantiation on 1 |
| 3. $R(\text{Ed})$ | hypothesis |
| 4. $P(\text{Ed})$ | modus ponens on 2 and 3 |
174. The two premises do not imply the conclusion.
175. Existential generalization.
176. (a) Assume p , prove q .
 (b) Assume $\neg q$, prove $\neg p$.
 (c) Assume $p \wedge \neg q$, show that this leads to a contradiction.
177. Direct proof: Assume p , show q . Indirect proof: Assume $\neg q$, show $\neg p$.
178. Suppose $x = 2k + 1$, $y = 2l$. Therefore $x + y = 2k + 1 + 2l = 2(k + l) + 1$, which is odd.
179. Suppose $n = 2k + 1$ but $n^2 = 2l$. Therefore $(2k + 1)^2 = 2l$, or $4k^2 + 4k + 1 = 2l$. Hence $2(2k^2 + 2k - l) = -1$ (even = odd), a contradiction. Therefore n^2 is odd.
180. Let $x = 2k + 1$, $y = 2l + 1$. Therefore $x + y = 2k + 1 + 2l + 1 = 2(k + l + 1)$, which is even.
181. Suppose $x = 2k + 1$ and $y = 2l + 1$, but $x + y = 2m + 1$. Therefore $(2k + 1) + (2l + 1) = 2m + 1$. Hence $2(k + l - m + 1) = 1$ (even = odd), which is a contradiction. Therefore $x + y$ is even.
182. Suppose $x = 2k$ and $y = 2l$, but $xy = 2m + 1$. Therefore $2k \cdot 2l = 2m + 1$. Hence $2(2kl - m) = 1$ (even = odd), which is a contradiction. Therefore xy is even.
183. Let $x = 2k + 1$. Therefore $x + 2 = 2k + 1 + 2 = 2(k + 1) + 1$, which is odd.
184. Suppose $x + 2 = 2k$. Therefore $x = 2k - 2 = 2(k - 1)$, which is even.

185. Suppose x is odd but $x + 2$ is even. Therefore $x = 2k + 1$ and $x + 2 = 2l$. Hence $(2k + 1) + 2 = 2l$. Therefore $2(k + 1 - l) = -1$ (even = odd), a contradiction.
186. Let $n = 2k$. Therefore $n + 1 = 2k + 1$, which is odd.
187. Suppose $n + 1$ is even. Therefore $n + 1 = 2k$. Therefore $n = 2k - 1 = 2(k - 1) + 1$, which is odd.
188. Suppose $n = 2k$ but $n + 1 = 2l$. Therefore $2k + 1 = 2l$ (even = odd), which is a contradiction.
189. If n is even, then $n = 2k$. Therefore $3n^2 + 8 = 3(2k)^2 + 8 = 12k^2 + 8 = 2(6k^2 + 4)$, which is even. If n is odd, then $n = 2k + 1$. Therefore $3n^2 + 8 = 3(2k + 1)^2 + 8 = 12k^2 + 12k + 11 = 2(6k^2 + 6k + 5) + 1$, which is odd.
190. If n is even, then $n^2 = (2k)^2 = 2(2k^2)$, which is even. If n is odd, then $n^2 = (2k + 1)^2 = 2(2k^2 + 2k) + 1$, which is odd.
191. If $m = 2k$ and $n = 2l$, then $mn = 4kl$. Hence mn is a multiple of 4.
192. False: $x = 2$ $y = 1/2$.
193. False: $x = 1/2$.
194. False: $x = 3/2$, $y = 3/2$.
195. Case 1, $x \geq 0$: then $x = |x|$, so $x \leq |x|$. Case 2, $x < 0$: here $x < 0$ and $0 < |x|$, so $x < |x|$.
196. It is easier to give a contraposition proof; it is usually easier to proceed from a simple expression (such as n) to a more complex expression (such as $3n + 5$ is even). Begin by supposing that n is not odd. Therefore n is even and hence $n = 2k$ for some integer k . Therefore $3n + 5 = 3(2k) + 5 = 6k + 5 = 2(3k + 2) + 1$, which is not even. If we try a direct proof, we assume that $3n + 5$ is even; that is, $3n + 5 = 2k$ for some integer k . From this we obtain $n = (2k - 5)/3$, and it is not obvious from this form that n is even.
197. Prove that (a) and (b) are equivalent and that (a) and (c) are equivalent.
198. If at most three people were born in each of the 12 months of the year, there would be at most 36 people.
199. Give a proof by cases. There are only six cases that need to be considered: $x = y = 1$; $x = 1, y = 2$; $x = 1, y = 3$; $x = 2, y = 1$; $x = y = 2$; $x = 2, y = 3$.
200. The steps in the “proof” cannot be reversed. Knowing that the squares of two numbers, -3 and 3 , are equal does not allow us to infer that the two numbers are equal.

Questions for Chapter 2

For each of the pairs of sets in 1-3 determine whether the first is a subset of the second, the second is a subset of the first, or neither is a subset of the other.

- The set of people who were born in the U.S., the set of people who are U.S. citizens.
- The set of students studying a programming language, the set of students studying Java.
- The set of animals living in the ocean, the set of fish.
- Prove or disprove: $A - (B \cap C) = (A - B) \cup (A - C)$.
- Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$ by giving a containment proof (that is, prove that the left side is a subset of the right side and that the right side is a subset of the left side).
- Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$ by giving an element table proof.
- Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$ by giving a proof using logical equivalence.

8. Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$ by giving a Venn diagram proof.
9. Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ by giving a containment proof (that is, prove that the left side is a subset of the right side and that the right side is a subset of the left side).
10. Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ by giving an element table proof.
11. Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ by giving a proof using logical equivalence.
12. Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ by giving a Venn diagram proof.
13. Prove or disprove: if A , B , and C are sets, then $A - (B \cap C) = (A - B) \cap (A - C)$.
14. Prove or disprove $A \oplus (B \oplus C) = (A \oplus B) \oplus C$.

In questions 15–18 use a Venn diagram to determine which relationship, \subseteq , $=$, or \supseteq , is true for the pair of sets.

15. $A \cup B$, $A \cup (B - A)$.
16. $A \cup (B \cap C)$, $(A \cup B) \cap C$.
17. $(A - B) \cup (A - C)$, $A - (B \cap C)$.
18. $(A - C) - (B - C)$, $A - B$.

In questions 19–23 determine whether the given set is the power set of some set. If the set is a power set, give the set of which it is a power set.

19. $\{\emptyset, \{\emptyset\}, \{a\}, \{\{a\}\}, \{\{\{a\}\}\}, \{\emptyset, a\}, \{\emptyset, \{a\}\}, \{\emptyset, \{\{a\}\}\}, \{a, \{a\}\}, \{a, \{\{a\}\}\}, \{\{a\}, \{\{a\}\}\}, \{\emptyset, a, \{a\}\}, \{\emptyset, a, \{\{a\}\}\}, \{\emptyset, \{a\}, \{\{a\}\}\}, \{a, \{a\}, \{\{a\}\}\}, \{\emptyset, a, \{a\}, \{\{a\}\}\}$.
20. $\{\emptyset, \{a\}\}$.
21. $\{\emptyset, \{a\}, \{\emptyset, a\}\}$.
22. $\{\emptyset, \{a\}, \{\emptyset\}, \{a, \emptyset\}\}$.
23. $\{\emptyset, \{a, \emptyset\}\}$.
24. Prove that $\overline{S \cup T} = S \cap T$ for all sets S and T .

In 25–35 mark each statement TRUE or FALSE. Assume that the statement applies to all sets.

25. $A - (B - C) = (A - B) - C$.
26. $(A - C) - (B - C) = A - B$.
27. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
28. $A \cap (B \cup C) = (A \cup B) \cap (A \cup C)$.
29. $\overline{A \cup B} \cup \overline{A} = \overline{A}$.
30. If $A \cup C = B \cup C$, then $A = B$.
31. If $A \cap C = B \cap C$, then $A = B$.
32. If $A \cap B = A \cup B$, then $A = B$.
33. If $A \oplus B = A$, then $B = A$.
34. There is a set A such that $|\mathcal{P}(A)| = 12$.

35. $A \oplus A = A$.
36. Find three subsets of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ such that the intersection of any two has size 2 and the intersection of all three has size 1.
37. Find $\bigcup_{i=1}^{+\infty} [-1/i, 1/i]$.
38. Find $\bigcap_{i=1}^{+\infty} (1 - \frac{1}{i}, 1)$.
39. Find $\bigcap_{i=1}^{+\infty} [1 - \frac{1}{i}, 1]$.
40. Find $\bigcap_{i=1}^{+\infty} (i, \infty)$.
41. Suppose $U = \{1, 2, \dots, 9\}$, $A =$ all multiples of 2, $B =$ all multiples of 3, and $C = \{3, 4, 5, 6, 7\}$. Find $C - (B - A)$.
42. Suppose $S = \{1, 2, 3, 4, 5\}$. Find $|\mathcal{P}(S)|$.

In questions 43–46 suppose $A = \{x, y\}$ and $B = \{x, \{x\}\}$. Mark the statement TRUE or FALSE.

43. $x \subseteq B$.
44. $\emptyset \in \mathcal{P}(B)$.
45. $\{x\} \subseteq A - B$.
46. $|\mathcal{P}(A)| = 4$.

In questions 47–54 suppose $A = \{a, b, c\}$. Mark the statement TRUE or FALSE.

47. $\{b, c\} \in \mathcal{P}(A)$.
48. $\{\{a\}\} \subseteq \mathcal{P}(A)$.
49. $\emptyset \subseteq A$.
50. $\{\emptyset\} \subseteq \mathcal{P}(A)$.
51. $\emptyset \subseteq A \times A$.
52. $\{a, c\} \in A$.
53. $\{a, b\} \in A \times A$.
54. $(c, c) \in A \times A$.

In questions 55–62 suppose $A = \{1, 2, 3, 4, 5\}$. Mark the statement TRUE or FALSE.

55. $\{1\} \in \mathcal{P}(A)$.
56. $\{\{3\}\} \subseteq \mathcal{P}(A)$.
57. $\emptyset \subseteq A$.
58. $\{\emptyset\} \subseteq \mathcal{P}(A)$.
59. $\emptyset \subseteq \mathcal{P}(A)$.
60. $\{2, 4\} \in A \times A$.

61. $\{\emptyset\} \in \mathcal{P}(A)$.

62. $(1, 1) \in A \times A$.

In questions 63–65 suppose the following are fuzzy sets:

$$F = \{0.7 \text{ Ann}, 0.1 \text{ Bill}, 0.8 \text{ Fran}, 0.3 \text{ Olive}, 0.5 \text{ Tom}\},$$

$$R = \{0.4 \text{ Ann}, 0.9 \text{ Bill}, 0.9 \text{ Fran}, 0.6 \text{ Olive}, 0.7 \text{ Tom}\}.$$

63. Find \overline{F} and \overline{R} .

64. Find $F \cup R$.

65. Find $F \cap R$.

In questions 66–75, suppose $A = \{a, b, c\}$ and $B = \{b, \{c\}\}$. Mark the statement TRUE or FALSE.

66. $c \in A - B$.

67. $|\mathcal{P}(A \times B)| = 64$.

68. $\emptyset \in \mathcal{P}(B)$.

69. $B \subseteq A$.

70. $\{c\} \subseteq B$.

71. $\{a, b\} \in A \times A$.

72. $\{b, c\} \in \mathcal{P}(A)$.

73. $\{b, \{c\}\} \in \mathcal{P}(B)$.

74. $\emptyset \subseteq A \times A$.

75. $\{\{\{c\}\}\} \subseteq \mathcal{P}(B)$.

76. Find A^2 if $A = \{1, a\}$.

In questions 77–89 determine whether the set is finite or infinite. If the set is finite, find its size.

77. $\{x \mid x \in \mathbf{Z} \text{ and } x^2 < 10\}$.

78. $\mathcal{P}(\{a, b, c, d\})$, where \mathcal{P} denotes the power set.

79. $\{1, 3, 5, 7, \dots\}$.

80. $A \times B$, where $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 2, 3\}$.

81. $\{x \mid x \in \mathbf{N} \text{ and } 9x^2 - 1 = 0\}$.

82. $\mathcal{P}(A)$, where A is the power set of $\{a, b, c\}$.

83. $A \times B$, where $A = \{a, b, c\}$ and $B = \emptyset$.

84. $\{x \mid x \in \mathbf{N} \text{ and } 4x^2 - 8 = 0\}$.

85. $\{x \mid x \in \mathbf{Z} \text{ and } x^2 = 2\}$.

86. $\mathcal{P}(A)$, where $A = \mathcal{P}(\{1, 2\})$.

87. $\{1, 10, 100, 1000, \dots\}$.

88. $S \times T$, where $S = \{a, b, c\}$ and $T = \{1, 2, 3, 4, 5\}$.

89. $\{x \mid x \in \mathbf{Z} \text{ and } x^2 < 8\}$.
90. Prove that between every two rational numbers a/b and c/d
 (a) there is a rational number. (b) there are an infinite number of rational numbers.
91. Prove that there is no smallest positive rational number.
92. Consider these functions from the set of licensed drivers in the state of New York. Is a function one-to-one if it assigns to a licensed driver his or her
 (a) birthdate
 (b) mother's first name
 (c) drivers license number?

In 93–94 determine whether each of the following sets is countable or uncountable. For those that are countably infinite exhibit a one-to-one correspondence between the set of positive integers and that set.

93. The set of positive rational numbers that can be written with denominators less than 3.
94. The set of irrational numbers between $\sqrt{2}$ and $\pi/2$.
95. Adapt the Cantor diagonalization argument to show that the set of positive real numbers less than 1 with decimal representations consisting only of 0s and 1s is uncountable.
96. Show that $(0, 1)$ has the same cardinality as $(0, 2)$.
97. Show that $(0, 1]$ and \mathbf{R} have the same cardinality.

In questions 98–106 determine whether the rule describes a function with the given domain and codomain.

98. $f: \mathbf{N} \rightarrow \mathbf{N}$ where $f(n) = \sqrt{n}$.
99. $h: \mathbf{R} \rightarrow \mathbf{R}$ where $h(x) = \sqrt{x}$.
100. $g: \mathbf{N} \rightarrow \mathbf{N}$ where $g(n) = \text{any integer } > n$.
101. $F: \mathbf{R} \rightarrow \mathbf{R}$ where $F(x) = \frac{1}{x-5}$.
102. $F: \mathbf{Z} \rightarrow \mathbf{R}$ where $F(x) = \frac{1}{x^2-5}$.
103. $F: \mathbf{Z} \rightarrow \mathbf{Z}$ where $F(x) = \frac{1}{x^2-5}$.
104. $G: \mathbf{R} \rightarrow \mathbf{R}$ where $G(x) = \begin{cases} x+2 & \text{if } x \geq 0 \\ x-1 & \text{if } x \leq 4. \end{cases}$
105. $f: \mathbf{R} \rightarrow \mathbf{R}$ where $f(x) = \begin{cases} x^2 & \text{if } x \leq 2 \\ x-1 & \text{if } x \geq 4. \end{cases}$
106. $G: \mathbf{Q} \rightarrow \mathbf{Q}$ where $G(p/q) = q$.
107. Give an example of a function $f: \mathbf{Z} \rightarrow \mathbf{Z}$ that is 1-1 and not onto \mathbf{Z} .
108. Give an example of a function $f: \mathbf{Z} \rightarrow \mathbf{Z}$ that is onto \mathbf{Z} but not 1-1.
109. Give an example of a function $f: \mathbf{Z} \rightarrow \mathbf{N}$ that is both 1-1 and onto \mathbf{N} .
110. Give an example of a function $f: \mathbf{N} \rightarrow \mathbf{Z}$ that is both 1-1 and onto \mathbf{Z} .
111. Give an example of a function $f: \mathbf{Z} \rightarrow \mathbf{N}$ that is 1-1 and not onto \mathbf{N} .

112. Give an example of a function $f: \mathbf{N} \rightarrow \mathbf{Z}$ that is onto \mathbf{Z} and not 1-1.
113. Suppose $f: \mathbf{N} \rightarrow \mathbf{N}$ has the rule $f(n) = 4n + 1$. Determine whether f is 1-1.
114. Suppose $f: \mathbf{N} \rightarrow \mathbf{N}$ has the rule $f(n) = 4n + 1$. Determine whether f is onto \mathbf{N} .
115. Suppose $f: \mathbf{Z} \rightarrow \mathbf{Z}$ has the rule $f(n) = 3n^2 - 1$. Determine whether f is 1-1.
116. Suppose $f: \mathbf{Z} \rightarrow \mathbf{Z}$ has the rule $f(n) = 3n - 1$. Determine whether f is onto \mathbf{Z} .
117. Suppose $f: \mathbf{N} \rightarrow \mathbf{N}$ has the rule $f(n) = 3n^2 - 1$. Determine whether f is 1-1.
118. Suppose $f: \mathbf{N} \rightarrow \mathbf{N}$ has the rule $f(n) = 4n^2 + 1$. Determine whether f is onto \mathbf{N} .
119. Suppose $f: \mathbf{R} \rightarrow \mathbf{R}$ where $f(x) = \lfloor x/2 \rfloor$.
 (a) Draw the graph of f . (b) Is f 1-1? (c) Is f onto \mathbf{R} ?
120. Suppose $f: \mathbf{R} \rightarrow \mathbf{R}$ where $f(x) = \lfloor x/2 \rfloor$.
 (a) If $S = \{x \mid 1 \leq x \leq 6\}$, find $f(S)$. (b) If $T = \{3, 4, 5\}$, find $f^{-1}(T)$.
121. Determine whether f is a function from the set of all bit strings to the set of integers if $f(S)$ is the position of a 1 bit in the bit string S .
122. Determine whether f is a function from the set of all bit strings to the set of integers if $f(S)$ is the number of 0 bits in S .
123. Determine whether f is a function from the set of all bit strings to the set of integers if $f(S)$ is the largest integer i such that the i th bit of S is 0 and $f(S) = 1$ when S is the empty string (the string with no bits).
124. Let $f(x) = \lfloor x^3/3 \rfloor$. Find $f(S)$ if S is:
 (a) $\{-2, -1, 0, 1, 2, 3\}$. (b) $\{0, 1, 2, 3, 4, 5\}$. (c) $\{1, 5, 7, 11\}$. (d) $\{2, 6, 10, 14\}$.
125. Suppose $f: \mathbf{R} \rightarrow \mathbf{Z}$ where $f(x) = \lceil 2x - 1 \rceil$.
 (a) Draw the graph of f . (b) Is f 1-1? (Explain) (c) Is f onto \mathbf{Z} ? (Explain)
126. Suppose $f: \mathbf{R} \rightarrow \mathbf{Z}$ where $f(x) = \lceil 2x - 1 \rceil$.
 (a) If $A = \{x \mid 1 \leq x \leq 4\}$, find $f(A)$. (b) If $B = \{3, 4, 5, 6, 7\}$, find $f(B)$.
 (c) If $C = \{-9, -8\}$, find $f^{-1}(C)$. (d) If $D = \{0.4, 0.5, 0.6\}$, find $f^{-1}(D)$.
127. Suppose $g: \mathbf{R} \rightarrow \mathbf{R}$ where $g(x) = \left\lfloor \frac{x-1}{2} \right\rfloor$.
 (a) Draw the graph of g . (b) Is g 1-1? (c) Is g onto \mathbf{R} ?
128. Suppose $g: \mathbf{R} \rightarrow \mathbf{R}$ where $g(x) = \left\lfloor \frac{x-1}{2} \right\rfloor$.
 (a) If $S = \{x \mid 1 \leq x \leq 6\}$, find $g(S)$. (b) If $T = \{2\}$, find $g^{-1}(T)$.
129. Show that $\lceil x \rceil = -\lfloor -x \rfloor$.
130. Prove or disprove: For all positive real numbers x and y , $\lfloor x \cdot y \rfloor \leq \lfloor x \rfloor \cdot \lfloor y \rfloor$.
131. Prove or disprove: For all positive real numbers x and y , $\lceil x \cdot y \rceil \leq \lceil x \rceil \cdot \lceil y \rceil$.
132. Suppose $g: A \rightarrow B$ and $f: B \rightarrow C$ where $A = \{1, 2, 3, 4\}$, $B = \{a, b, c\}$, $C = \{2, 7, 10\}$, and f and g are defined by $g = \{(1, b), (2, a), (3, a), (4, b)\}$ and $f = \{(a, 10), (b, 7), (c, 2)\}$. Find $f \circ g$.
133. Suppose $g: A \rightarrow B$ and $f: B \rightarrow C$ where $A = \{1, 2, 3, 4\}$, $B = \{a, b, c\}$, $C = \{2, 7, 10\}$, and f and g are defined by $g = \{(1, b), (2, a), (3, a), (4, b)\}$ and $f = \{(a, 10), (b, 7), (c, 2)\}$. Find f^{-1} .

In questions 134–137 suppose that $g: A \rightarrow B$ and $f: B \rightarrow C$ where $A = B = C = \{1, 2, 3, 4\}$, $g = \{(1, 4), (2, 1), (3, 1), (4, 2)\}$, and $f = \{(1, 3), (2, 2), (3, 4), (4, 2)\}$.

134. Find $f \circ g$.

135. Find $g \circ f$.

136. Find $g \circ g$.

137. Find $g \circ (g \circ g)$.

In questions 138–141 suppose $g: A \rightarrow B$ and $f: B \rightarrow C$ where $A = \{1, 2, 3, 4\}$, $B = \{a, b, c\}$, $C = \{2, 8, 10\}$, and g and f are defined by $g = \{(1, b), (2, a), (3, b), (4, a)\}$ and $f = \{(a, 8), (b, 10), (c, 2)\}$.

138. Find $f \circ g$.

139. Find f^{-1} .

140. Find $f \circ f^{-1}$.

141. Explain why g^{-1} is not a function.

In questions 142–143 suppose $g: A \rightarrow B$ and $f: B \rightarrow C$ where $A = \{a, b, c, d\}$, $B = \{1, 2, 3\}$, $C = \{2, 3, 6, 8\}$, and g and f are defined by $g = \{(a, 2), (b, 1), (c, 3), (d, 2)\}$ and $f = \{(1, 8), (2, 3), (3, 2)\}$.

142. Find $f \circ g$.

143. Find f^{-1} .

144. For any function $f: A \rightarrow B$, define a new function $g: \mathcal{P}(A) \rightarrow \mathcal{P}(B)$ as follows: for every $S \subseteq A$, $g(S) = \{f(x) \mid x \in S\}$. Prove that f is onto if and only if g is onto.

In questions 145–149 find the inverse of the function f or else explain why the function has no inverse.

145. $f: \mathbf{Z} \rightarrow \mathbf{Z}$ where $f(x) = x \bmod 10$.

146. $f: A \rightarrow B$ where $A = \{a, b, c\}$, $B = \{1, 2, 3\}$ and $f = \{(a, 2), (b, 1), (c, 3)\}$.

147. $f: \mathbf{R} \rightarrow \mathbf{R}$ where $f(x) = 3x - 5$.

148. $f: \mathbf{R} \rightarrow \mathbf{R}$ where $f(x) = \lfloor 2x \rfloor$.

149. $f: \mathbf{Z} \rightarrow \mathbf{Z}$ where $f(x) = \begin{cases} x - 2 & \text{if } x \geq 5 \\ x + 1 & \text{if } x \leq 4. \end{cases}$

150. Suppose $g: A \rightarrow B$ and $f: B \rightarrow C$, where $f \circ g$ is 1-1 and g is 1-1. Must f be 1-1?

151. Suppose $g: A \rightarrow B$ and $f: B \rightarrow C$, where $f \circ g$ is 1-1 and f is 1-1. Must g be 1-1?

152. Suppose $f: \mathbf{R} \rightarrow \mathbf{R}$ and $g: \mathbf{R} \rightarrow \mathbf{R}$ where $g(x) = 2x + 1$ and $g \circ f(x) = 2x + 11$. Find the rule for f .

In questions 153–157 for each partial function, determine its domain, codomain, domain of definition, set of values for which it is undefined or if it is a total function:

153. $f: \mathbf{Z} \rightarrow \mathbf{R}$ where $f(n) = 1/n$.

154. $f: \mathbf{Z} \rightarrow \mathbf{Z}$ where $f(n) = \lceil n/2 \rceil$.

155. $f: \mathbf{Z} \times \mathbf{Z} \rightarrow \mathbf{Q}$ where $f(m, n) = m/n$.

156. $f: \mathbf{Z} \times \mathbf{Z} \rightarrow \mathbf{Z}$ where $f(m, n) = mn$.

157. $f: \mathbf{Z} \times \mathbf{Z} \rightarrow \mathbf{Z}$ where $f(m, n) = m - n$ if $m > n$.

- 158.** For the partial function $f: \mathbf{Z} \times \mathbf{Z} \rightarrow \mathbf{R}$ defined by $f(m, n) = \frac{1}{n^2 - m^2}$, determine its domain, codomain, domain of definition, and set of values for which it is undefined or whether it is a total function.
- 159.** Let $f: \{1, 2, 3, 4, 5\} \rightarrow \{1, 2, 3, 4, 5, 6\}$ be a function.
- How many total functions are there?
 - How many of these functions are one-to-one?

In questions 160–166 find a formula that generates the following sequence a_1, a_2, a_3, \dots

- 160.** 5, 9, 13, 17, 21, \dots
- 161.** 3, 3, 3, 3, 3, \dots
- 162.** 15, 20, 25, 30, 35, \dots
- 163.** 1, 0.9, 0.8, 0.7, 0.6, \dots
- 164.** 1, $1/3$, $1/5$, $1/7$, $1/9$, \dots
- 165.** 2, 0, 2, 0, 2, 0, 2, \dots
- 166.** 0, 2, 0, 2, 0, 2, 0, \dots

In questions 167–178, describe each sequence recursively. Include initial conditions and assume that the sequences begin with a_1 .

- 167.** $a_n = 5^n$.
- 168.** The Fibonacci numbers.
- 169.** 0, 1, 0, 1, 0, 1, \dots
- 170.** $a_n = 1 + 2 + 3 + \dots + n$.
- 171.** 3, 2, 1, 0, -1 , -2 , \dots
- 172.** $a_n = n!$.
- 173.** $1/2$, $1/3$, $1/4$, $1/5$, \dots
- 174.** 0.1, 0.11, 0.111, 0.1111, \dots
- 175.** $1^2, 2^2, 3^2, 4^2, \dots$
- 176.** 1, 111, 11111, 1111111, \dots
- 177.** $a_n =$ the number of subsets of a set of size n .
- 178.** 1, 101, 10101, 1010101, \dots
- 179.** Verify that $a_n = 6$ is a solution to the recurrence relation $a_n = 4a_{n-1} - 3a_{n-2}$.
- 180.** Verify that $a_n = 3^n$ is a solution to the recurrence relation $a_n = 4a_{n-1} - 3a_{n-2}$.
- 181.** Verify that $a_n = 3^{n+4}$ is a solution to the recurrence relation $a_n = 4a_{n-1} - 3a_{n-2}$.
- 182.** Verify that $a_n = 3^n + 1$ is a solution to the recurrence relation $a_n = 4a_{n-1} - 3a_{n-2}$.
- 183.** Verify that $a_n = 7 \cdot 3^n - \pi$ is a solution to the recurrence relation $a_n = 4a_{n-1} - 3a_{n-2}$.

In questions 184–188 find a recurrence relation with initial condition(s) satisfied by the sequence. Assume a_0 is the first term of the sequence.

184. $a_n = 2^n$.
185. $a_n = 2^n + 1$.
186. $a_n = (-1)^n$.
187. $a_n = 3n - 1$.
188. $a_n = \sqrt{2}$.
189. You take a job that pays \$25,000 annually.
- How much do you earn n years from now if you receive a three percent raise each year?
 - How much do you earn n years from now if you receive a five percent raise each year?
 - How much do you earn n years from now if each year you receive a raise of \$1000 plus two percent of your previous year's salary.
190. Suppose inflation continues at three percent annually. (That is, an item that costs \$1.00 now will cost \$1.03 next year.) Let a_n = the value (that is, the purchasing power) of one dollar after n years.
- Find a recurrence relation for a_n .
 - What is the value of \$1.00 after 20 years?
 - What is the value of \$1.00 after 80 years?
 - If inflation were to continue at ten percent annually, find the value of \$1.00 after 20 years.
 - If inflation were to continue at ten percent annually, find the value of \$1.00 after 80 years.
191. Find the sum $1/4 + 1/8 + 1/16 + 1/32 + \dots$.
192. Find the sum $2 + 4 + 8 + 16 + 32 + \dots + 2^{28}$.
193. Find the sum $2 - 4 + 8 - 16 + 32 - \dots - 2^{28}$.
194. Find the sum $1 - 1/2 + 1/4 - 1/8 + 1/16 - \dots$.
195. Find the sum $2 + 1/2 + 1/8 + 1/32 + 1/128 + \dots$.
196. Find the sum $112 + 113 + 114 + \dots + 673$.
197. Find $\sum_{i=1}^6 ((-2)^i - 2^i)$.
198. Find $\sum_{j=1}^3 \sum_{i=1}^j ij$.
199. Rewrite $\sum_{i=-3}^4 (i^2 + 1)$ so that the index of summation has lower limit 0 and upper limit 7.
200. Find a 2×2 matrix $\mathbf{A} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ such that $\mathbf{A}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.
201. Suppose \mathbf{A} is a 6×8 matrix, \mathbf{B} is an 8×5 matrix, and \mathbf{C} is a 5×9 matrix. Find the number of rows, the number of columns, and the number of entries in $\mathbf{A}(\mathbf{BC})$.
202. Let $\mathbf{A} = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$. Find \mathbf{A}^n where n is a positive integer.
203. Suppose $\mathbf{A} = \begin{pmatrix} 3 & 5 \\ 2 & 4 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 2 & 1 \\ 0 & 6 \end{pmatrix}$. Find a matrix \mathbf{B} such that $\mathbf{AB} = \mathbf{C}$ or prove that no such matrix exists.

204. Suppose $\mathbf{B} = \begin{pmatrix} 3 & 5 \\ 2 & 4 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 2 & 1 \\ 0 & 6 \end{pmatrix}$. Find a matrix \mathbf{A} such that $\mathbf{AB} = \mathbf{C}$ or prove that no such matrix exists.

205. Suppose $\mathbf{B} = \begin{pmatrix} 6 & 2 \\ 3 & 1 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 2 & 1 \\ 0 & 6 \end{pmatrix}$. Find a matrix \mathbf{A} such that $\mathbf{AB} = \mathbf{C}$ or prove that no such matrix exists.

In questions 206–212 determine whether the statement is true or false.

206. If $\mathbf{AB} = \mathbf{AC}$, then $\mathbf{B} = \mathbf{C}$.

207. If $\mathbf{A} = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$, then $\mathbf{A}^{-1} = \begin{pmatrix} 2 & 5 \\ 1 & -3 \end{pmatrix}$.

208. If $\mathbf{A} = \begin{pmatrix} 1 & 3 \\ -5 & 2 \end{pmatrix}$, then $\mathbf{A}^2 = \begin{pmatrix} 1 & 9 \\ 25 & 4 \end{pmatrix}$.

209. If \mathbf{A} is a 6×4 matrix and \mathbf{B} is a 4×5 matrix, then \mathbf{AB} has 16 entries.

210. If \mathbf{A} and \mathbf{B} are 2×2 matrices such that $\mathbf{AB} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, then $\mathbf{A} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ or $\mathbf{B} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

211. If \mathbf{A} and \mathbf{B} are 2×2 matrices, then $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$.

212. $\mathbf{AB} = \mathbf{BA}$ for all 2×2 matrices \mathbf{A} and \mathbf{B} .

213. Suppose $\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$. Find

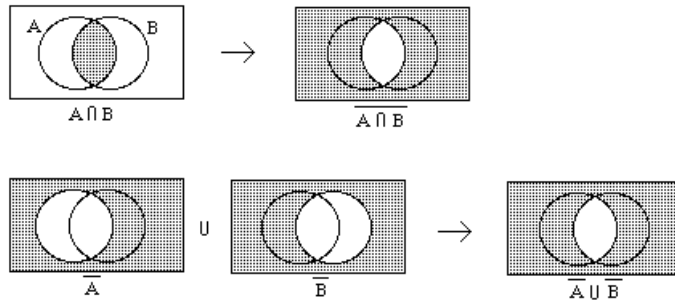
- (a) the join of \mathbf{A} and \mathbf{B} . (b) the meet of \mathbf{A} and \mathbf{B} . (c) the Boolean product of \mathbf{A} and \mathbf{B} .

214. Suppose \mathbf{A} is a 2×2 matrix with real number entries such that $\mathbf{AB} = \mathbf{BA}$ for all 2×2 matrices. What relationships must exist among the entries of \mathbf{A} ?

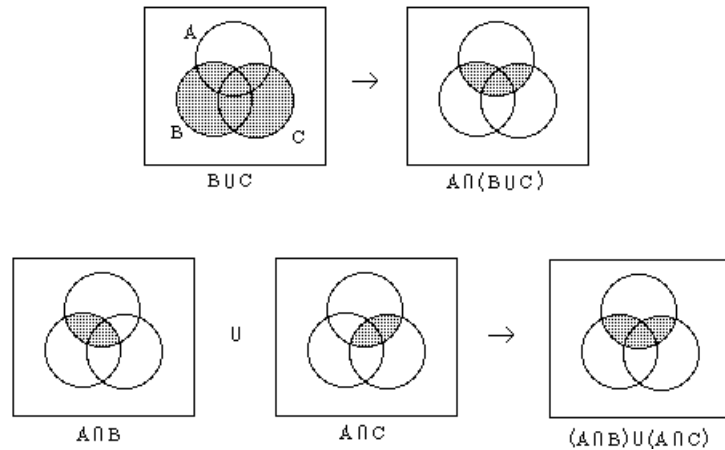
Answers for Chapter 2

1. The first is a subset of the second, but the second is not a subset of the first.
2. The second is a subset of the first, but the first is not a subset of the second.
3. Neither is a subset of the other.
4. True, since $A - (B \cap C) = A \cap \overline{B \cap C} = A \cap (\overline{B} \cup \overline{C}) = (A \cap \overline{B}) \cup (A \cap \overline{C}) = (A - B) \cup (A - C)$.
5. $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$: Let $x \in \overline{A \cap B}$. $\therefore x \notin A \cap B$, $\therefore x \notin A$ or $x \notin B$, $\therefore x \in \overline{A}$ or $x \in \overline{B}$, $\therefore x \in \overline{A} \cup \overline{B}$. Reversing the steps shows that $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$.
6. The columns for $\overline{A \cap B}$ and $\overline{A} \cup \overline{B}$ match: each entry is 0 if and only if A and B have the value 1.
7. $\overline{A \cap B} = \{x \mid x \in \overline{A \cap B}\} = \{x \mid x \notin A \cap B\} = \{x \mid \neg(x \in A \cap B)\} = \{x \mid \neg(x \in A \wedge x \in B)\} = \{x \mid \neg(x \in A) \vee \neg(x \in B)\} = \{x \mid x \notin A \vee x \notin B\} = \{x \mid x \in \overline{A} \vee x \in \overline{B}\} = \{x \mid x \in \overline{A} \cup \overline{B}\} = \overline{A} \cup \overline{B}$.

8.



9. $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$: Let $x \in A \cap (B \cup C)$. $\therefore x \in A$ and $x \in B \cup C$, $\therefore x \in A$ and $x \in B$, or $x \in A$ and $x \in C$, $\therefore x \in (A \cap B) \cup (A \cap C)$. Reversing the steps gives the opposite containment.
10. Each set has the same values in the element table: the value is 1 if and only if A has the value 1 and either B or C has the value 1.
11. $A \cap (B \cup C) = \{x \mid x \in A \cap (B \cup C)\} = \{x \mid x \in A \wedge x \in (B \cup C)\} = \{x \mid x \in A \wedge (x \in B \vee x \in C)\} = \{x \mid (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C)\} = \{x \mid x \in A \cap B \vee x \in A \cap C\} = \{x \mid x \in (A \cap B) \cup (A \cap C)\} = (A \cap B) \cup (A \cap C)$.
- 12.



13. False. For example, let $A = \{1, 2\}$, $B = \{1\}$, $C = \{2\}$.
14. True, using either a membership table or a containment proof, for example.
15. =.
16. \supseteq .
17. =.
18. \subseteq .
19. Yes $\{\emptyset, a, \{a\}, \{\{a\}\}\}$.
20. Yes, $\{a\}$.
21. No, it lacks $\{\emptyset\}$.
22. Yes, $\{\{a, \emptyset\}\}$.
23. No, it lacks $\{a\}$ and $\{\emptyset\}$.
24. Since $\overline{S \cup T} = \overline{S} \cap \overline{T}$ (De Morgan's law), the complements are equal.
25. False.
26. False.
27. True.
28. False.
29. True.
30. False.

31. False.
32. True.
33. False.
34. False.
35. False.
36. For example, $\{1, 2, 3\}$, $\{2, 3, 4\}$, $\{1, 3, 4\}$.
37. $[-1, 1]$.
38. \emptyset .
39. $\{1\}$.
40. \emptyset .
41. $\{4, 5, 6, 7\}$.
42. 32.
43. False.
44. True.
45. False.
46. True.
47. True.
48. True.
49. True.
50. True.
51. True.
52. True.
53. False.
54. True.
55. True.
56. True.
57. True.
58. True.
59. True.
60. False.
61. False.
62. True.
63. $\bar{F} = \{0.3 \text{ Ann}, 0.9 \text{ Bill}, 0.2 \text{ Fran}, 0.7 \text{ Olive}, 0.5 \text{ Tom}\}$,
 $\bar{R} = \{0.6 \text{ Ann}, 0.1 \text{ Bill}, 0.1 \text{ Fran}, 0.4 \text{ Olive}, 0.3 \text{ Tom}\}$
64. $\{0.7 \text{ Ann}, 0.9 \text{ Bill}, 0.9 \text{ Fran}, 0.6 \text{ Olive}, 0.7 \text{ Tom}\}$.
65. $\{0.4 \text{ Ann}, 0.1 \text{ Bill}, 0.8 \text{ Fran}, 0.3 \text{ Olive}, 0.5 \text{ Tom}\}$.
66. True.
67. True.
68. True.
69. False.
70. False.
71. False.
72. True.

73. True.
74. True.
75. True.
76. $A^2 = \{(1, 1), (1, a), (a, 1), (a, a)\}$
77. 7.
78. 16.
79. Infinite.
80. 15.
81. 0.
82. 256.
83. 0.
84. 0.
85. 0.
86. 16.
87. Infinite.
88. 15.
89. 5.
90. (a) Assume $\frac{a}{b} < \frac{c}{d}$. Then $\frac{a}{b} < \frac{\frac{a}{b} + \frac{c}{d}}{2} = \frac{ad+bc}{2bd} < \frac{c}{d}$.
 (b) Assume $\frac{a}{b} < \frac{c}{d}$. Let m_1 be the midpoint of $[\frac{a}{b}, \frac{c}{d}]$. For $i > 1$ let m_i be the midpoint of $[\frac{a}{b}, m_{i-1}]$.
91. If $0 < \frac{a}{b}$, then $0 < \dots < \frac{a}{4b} < \frac{a}{3b} < \frac{a}{2b} < \frac{a}{b}$.
92. (a) No (b) No (c) Yes
93. Countable. To find a correspondence, follow the path in Example 4 in Section 2.5, using only the first three lines.
94. Uncountable
95. Assume that these numbers are countable, and list them in order r_1, r_2, r_3, \dots . Then form a new number r , whose i -th decimal digit is 0, if the i -th decimal digit of r_i is 1, and whose i -th decimal digit is 1, if the i -th decimal digit of r_i is 0. Clearly r is not in the list r_1, r_2, r_3, \dots , therefore the original assumption is false.
96. The function $f(x) = 2x$ is one-to-one and onto.
97. Example 2.5.6 shows that $|(0, 1]| = |(0, 1|]$, and Exercise 2.5.34 shows that $|(0, 1]| = \mathbf{R}$.
98. Not a function; $f(2)$ is not an integer.
99. Function.
100. Not a function; $g(1)$ has more than one value.
101. Not a function; $F(5)$ not defined.
102. Function.
103. Not a function; $F(1)$ not an integer.
104. Not a function; the cases overlap. For example, $G(1)$ is equal to both 3 and 0.
105. Not a function; $f(3)$ not defined.
106. Not a function; $f(1/2) = 2$ and $f(2/4) = 4$.
107. $f(n) = 2n$.
108. $f(n) = \lfloor n/2 \rfloor$.
109. $f(n) = \begin{cases} -2n, & n \leq 0 \\ 2n - 1, & n > 0. \end{cases}$

$$110. f(n) = \begin{cases} \frac{-n}{2}, & n \text{ even} \\ \frac{n+1}{2}, & n \text{ odd.} \end{cases}$$

$$111. f(n) = \begin{cases} -2n, & n \leq 0 \\ 2n+1, & n > 0. \end{cases}$$

$$112. f(n) = \begin{cases} \frac{-n}{2}, & n \text{ even} \\ \frac{n-1}{2}, & n \text{ odd.} \end{cases}$$

113. Yes.

114. No.

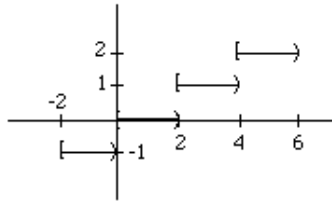
115. No.

116. No.

117. Yes.

118. No.

119. (a)



(b) No.

(c) No.

120. (a) $\{0, 1, 2, 3\}$

(b) $[6, 12)$.

121. No; there may be no 1 bits or more than one 1 bit.

122. Yes.

123. No; f not defined for the string of all 1's, for example $S = 11111$.

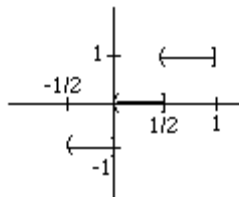
124. (a) $\{-3, -1, 0, 2, 9\}$.

(b) $\{0, 2, 9, 21, 41\}$.

(c) $\{0, 41, 114, 443\}$.

(d) $\{2, 72, 333, 914\}$.

125. (a)



(b) No.

(c) Yes.

126. (a) $\{1, 2, 3, 4, 5, 6, 7\}$.

(b) $\{5, 7, 9, 11, 13\}$.

(c) $(-9/2, -7/2]$.

(d) \emptyset .

127. (a)

- (b) No.
(c) No.
128. (a) $\{0, 1, 2\}$.
(b) $[5, 7)$.
129. Let $n = \lceil x \rceil$, so that $n - 1 < x \leq n$. Multiplying by -1 yields $-n + 1 > -x \geq -n$, which means that $-n = \lfloor -x \rfloor$.
130. False: $x = y = 1.5$.
131. True: $x \leq \lceil x \rceil$, $y \leq \lceil y \rceil$; therefore $xy \leq \lceil x \rceil \lceil y \rceil$; since $\lceil x \rceil \lceil y \rceil$ is an integer at least as great as xy , then $\lceil xy \rceil \leq \lceil x \rceil \lceil y \rceil$.
132. $\{(1, 7), (2, 10), (3, 10), (4, 7)\}$.
133. $\{(2, c), (7, b), (10, a)\}$.
134. $\{(1, 2), (2, 3), (3, 3), (4, 2)\}$.
135. $\{(1, 1), (2, 1), (3, 2), (4, 1)\}$.
136. $\{(1, 2), (2, 4), (3, 4), (4, 1)\}$.
137. $\{(1, 1), (2, 2), (3, 2), (4, 4)\}$.
138. $\{(1, 10), (2, 8), (3, 10), (4, 8)\}$.
139. $\{(2, c), (8, a), (10, b)\}$.
140. $\{(2, 2), (8, 8), (10, 10)\}$.
141. $g^{-1}(a)$ is equal to both 2 and 4.
142. $\{(a, 3), (b, 8), (c, 2), (d, 3)\}$.
143. $\{(2, 3), (3, 2), (8, 1)\}$.
144. Suppose f is onto. Let $T \in \mathcal{P}(B)$ and let $S = \{x \in A \mid f(x) \in T\}$. Then $g(S) = T$, and g is onto. If f is not onto B , let $y \in B - f(A)$. Then there is no subset S of A such that $g(S) = \{y\}$.
145. $f^{-1}(10)$ does not exist.
146. $\{(1, b), (2, a), (3, c)\}$.
147. $f^{-1}(x) = \frac{5+x}{3}$.
148. $f^{-1}(\frac{1}{2})$ does not exist.
149. $f^{-1}(5)$ is not a single value.
150. No.
151. Yes.
152. $f(x) = x + 5$.
153. $\mathbf{Z}, \mathbf{R}, \mathbf{Z} - \{0\}, \{0\}$.
154. $\mathbf{Z}, \mathbf{Z}, \mathbf{Z}$, total function.
155. $\mathbf{Z} \times \mathbf{Z}, \mathbf{Q}, \mathbf{Z} \times (\mathbf{Z} - \{0\}), \mathbf{Z} \times \{0\}$.
156. $\mathbf{Z} \times \mathbf{Z}, \mathbf{Z}, \mathbf{Z} \times \mathbf{Z}$, total function.
157. $\mathbf{Z} \times \mathbf{Z}, \mathbf{Z}, \{(m, n) \mid m > n\}, \{(m, n) \mid m \leq n\}$.
158. $\mathbf{Z} \times \mathbf{Z}, \mathbf{R}, \{(m, n) \mid m \neq n \text{ or } m \neq -n\}, \{(m, n) \mid m = n \text{ or } m = -n\}$.
159. (a) $6^5 = 7,776$.
(b) $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 = 720$.
160. $a_n = 4n + 1$.
161. $a_n = 3$.
162. $a_n = 5(n + 2)$.
163. $a_n = 1 - (n - 1)/10$.

164. $a_n = 1/(2n - 1)$.
165. $a_n = 1 + (-1)^{n+1}$.
166. $a_n = 1 + (-1)^n$.
167. $a_n = 5a_{n-1}, a_1 = 5$.
168. $a_n = a_{n-1} + a_{n-2}, a_1 = a_2 = 1$.
169. $a_n = a_{n-2}, a_1 = 0, a_2 = 1$.
170. $a_n = a_{n-1} + n, a_1 = 1$.
171. $a_n = a_{n-1} - 1, a_1 = 3$.
172. $a_n = na_{n-1}, a_1 = 1$.
173. $a_n = \frac{a_{n-1}}{1 + a_{n-1}}, a_1 = 1/2$.
174. $a_n = a_{n-1} + 1/10^n, a_1 = 0.1$.
175. $a_n = a_{n-1} + 2n - 1, a_1 = 1$.
176. $a_n = 100a_{n-1} + 11$.
177. $a_n = 2 \cdot a_{n-1}, a_1 = 2$.
178. $a_n = 100a_{n-1} + 1, a_1 = 1$.
179. $4 \cdot 6 - 3 \cdot 6 = 1 \cdot 6 = 6$.
180. $4 \cdot 3^{n-1} - 3 \cdot 3^{n-2} = 4 \cdot 3^{n-1} - 3^{n-1} = 3 \cdot 3^{n-1} = 3^n$.
181. $4 \cdot 3^{n+3} - 3 \cdot 3^{n+2} = 4 \cdot 3^{n+3} - 3^{n+3} = 3 \cdot 3^{n+3} = 3^{n+4}$.
182. $4(3^{n-1} + 1) - 3(3^{n-2} + 1) = 4 \cdot 3^{n-1} - 3^{n-1} + 4 - 3 = 3^{n-1}(4 - 1) + 1 = 3^n + 1$.
183. $4(7 \cdot 3^{n-1} - \pi) - 3(7 \cdot 3^{n-2} - \pi) = 28 \cdot 3^{n-1} - 7 \cdot 3^{n-1} - 4\pi + 3\pi = 7 \cdot 3^n - \pi$.
184. $a_n = 2a_{n-1}, a_0 = 1$.
185. $a_n = 2a_{n-1} - 1, a_0 = 2$.
186. $a_n = -a_{n-1}, a_0 = 1$.
187. $a_n = a_{n-1} + 3, a_0 = -1$.
188. $a_n = a_{n-1}, a_0 = \sqrt{2}$.
189. (a) $25,000 \cdot 1.03^n$. (b) $25,000 \cdot 1.05^n$. (c) $25,000 \cdot 1.02^n + 1,000 \left(\frac{1.02^n - 1}{0.02} \right)$.
190. (a) $a_n = a_{n-1}/1.03$. (b) $a_{20} = 1/1.03^{20} \approx 0.55$. (c) $a_{80} = 1/1.03^{80} \approx 0.09$. (d) $1/1.1^{20} \approx 0.15$. (e) $1/1.1^{80} \approx 0.00$.
191. $1/2$.
192. $2^{29} - 2$.
193. $\frac{2}{3} + \frac{2}{3}(2^{29})$.
194. $2/3$.
195. $8/3$.
196. 220,585.
197. -84.
198. 25.
199. $\sum_{i=0}^7 ((i - 3)^2 + 1)$.
200. A matrix of the form $\begin{pmatrix} -2a & a \\ -4a & 2a \end{pmatrix}$ where $a \neq 0$.
201. $\mathbf{A}(\mathbf{BC})$ has 6 rows, 9 columns, and 54 entries.
202. $\mathbf{A}^n = \begin{pmatrix} 1 & mn \\ 0 & 1 \end{pmatrix}$.

203. $\begin{pmatrix} 4 & -13 \\ -2 & 8 \end{pmatrix}$.

204. $\begin{pmatrix} 3 & -7/2 \\ -6 & 9 \end{pmatrix}$.

205. None exists since $\det \mathbf{B} = 0$ and $\det \mathbf{C} \neq 0$.

206. False.

207. False.

208. False.

209. False.

210. False.

211. True.

212. False.

213. (a) $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. (b) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$. (c) $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$.

214. $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$.

Questions for Chapter 3

- Describe an algorithm that takes a list of n integers a_1, a_2, \dots, a_n and finds the number of integers each greater than five in the list.
- Describe an algorithm that takes a list of integers a_1, a_2, \dots, a_n ($n \geq 2$) and finds the second-largest integer in the sequence by going through the list and keeping track of the largest and second-largest integer encountered.
- Describe an algorithm that takes a list of n integers ($n \geq 1$) and finds the location of the last even integer in the list, and returns 0 if there are no even integers in the list.
- Describe an algorithm that takes a list of n integers ($n \geq 1$) and finds the average of the largest and smallest integers in the list.
- Express a brute-force algorithm that finds the second largest element in a list a_1, a_2, \dots, a_n ($n \geq 2$) of distinct integers by finding the largest element, placing it at the beginning of the sequence, then finding the largest element of the remaining sequence.
- Express a brute-force algorithm that finds the largest product of two numbers in a list a_1, a_2, \dots, a_n ($n \geq 2$) that is less than a threshold N .
- Describe in words how the binary search works.
- Show how the binary search algorithm searches for 27 in the following list: 5 6 8 12 15 21 25 31.
- You have supplies of boards that are one foot, five feet, seven feet, and twelve feet long. You need to lay pieces end-to-end to make a molding 15 feet long and wish to do this using the fewest number of pieces possible. Explain why the greedy algorithm of taking boards of the longest length at each stage (so long as the total length of the boards selected does not exceed 15 feet) does not give the fewest number of boards possible.
- Prove or disprove that the greedy algorithm for making change always uses the fewest coins possible when the denominations available are pennies (1-cent coins), nickels (5-cent coins), and quarters (25-cent coins).

11. Prove or disprove that the greedy algorithm for making change always uses the fewest coins possible when the denominations available are 1-cent coins, 8-cent coins, and 20-cent coins.
12. Use the definition of big- O to prove that $1^2 + 2^2 + \cdots + n^2$ is $O(n^3)$.
13. Use the definition of big- O to prove that $\frac{3n - 8 - 4n^3}{2n - 1}$ is $O(n^2)$.
14. Use the definition of big- O to prove that $1^3 + 2^3 + \cdots + n^3$ is $O(n^4)$.
15. Use the definition of big- O to prove that $\frac{6n + 4n^5 - 4}{7n^2 - 3}$ is $O(n^3)$.
16. Use the definition of big- O to prove that $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + (n - 1) \cdot n$ is $O(n^3)$.
17. Let $f(n) = 3n^2 + 8n + 7$. Show that $f(n)$ is $O(n^2)$. Find C and k from the definition.

In questions 18–23 find the best big- O function for the function. Choose your answer from among the following:

$1, \log_2 n, n, n \log_2 n, n^2, n^3, \dots, 2^n, n!$

18. $f(n) = 1 + 4 + 7 + \cdots + (3n + 1)$.
19. $g(n) = 1 + 3 + 5 + 7 + \cdots + (2n - 1)$.
20. $\frac{3 - 2n^4 - 4n}{2n^3 - 3n}$.
21. $f(n) = 1 + 2 + 3 + \cdots + (n^2 - 1) + n^2$.
22. $\lceil n + 2 \rceil \cdot \lceil n/3 \rceil$.
23. $3n^4 + \log_2 n^8$.
24. Show that $\sum_{j=1}^n (j^3 + j)$ is $O(n^4)$.
25. Show that $f(x) = (x + 2) \log_2(x^2 + 1) + \log_2(x^3 + 1)$ is $O(x \log_2 x)$.
26. Find the best big- O function for $n^3 + \sin n^7$.
27. Find the best big- O function for $\frac{x^3 + 7x}{3x + 1}$.
28. Prove that $5x^4 + 2x^3 - 1$ is $\Theta(x^4)$.
29. Prove that $\frac{x^3 + 7x^2 + 3}{2x + 1}$ is $\Theta(x^2)$.
30. Prove that $x^3 + 7x + 2$ is $\Omega(x^3)$.
31. Arrange the functions $n^{3/2}$, $\log(n^n)$, $(n^{100})^n$ and $\log(n!)$ in a list so that each function is big- O of the next function.
32. Arrange the following functions in a list so each is big- O of the next one in the list: $n^3 + 88n^2 + 3$, $\log n^4$, 3^n , $n^2 \log n$, $n \cdot 2^n$, 10000
33. Arrange the following functions in a list so each is big- O of the next one in the list: $\log n^2$, $\log \log n$, $n \log n$, $\log(n^2 + 1)$, $\log 2^n$
34. Find all pairs of functions in this list that are of the same order: $n^2 + \log n$, $2^n + 3^n$, $100n^3 + n^2$, $n^2 + 2^n$, $n^2 + n^3$, $3n^3 + 2^n$.
35. Suppose you have two different algorithms for solving a problem. To solve a problem of size n , the first algorithm uses exactly $n\sqrt{n}$ operations and the second algorithm uses exactly $n^2 \log n$ operations. As n grows, which algorithm uses fewer operations?

In questions 36–46 find the “best” big- O notation to describe the complexity of the algorithm. Choose your answers from the following:

$$1, \log_2 n, n, n \log_2 n, n^2, n^3, \dots, 2^n, n!.$$

36. A binary search of n elements.
37. A linear search to find the smallest number in a list of n numbers.
38. An algorithm that lists all ways to put the numbers $1, 2, 3, \dots, n$ in a row.
39. An algorithm that prints all bit strings of length n .
40. The number of print statements in the following:
- ```

i := 1, j := 1
while i ≤ n
 while j ≤ i
 print “hello”;
 j := j + 1
 i := i + 1

```
41. The number of print statements in the following:
- ```

while n > 1
  print “hello”;
  n := ⌊n/2⌋

```
42. An iterative algorithm to compute $n!$, (counting the number of multiplications).
43. An algorithm that finds the average of n numbers by adding them and dividing by n .
44. An algorithm that prints all subsets of size three of the set $\{1, 2, 3, \dots, n\}$.
45. The best-case analysis of a linear search of a list of size n (counting the number of comparisons).
46. The worst-case analysis of a linear search of a list of size n (counting the number of comparisons).
47. Give a big- O estimate for the number of operations (where an operation is an addition or a multiplication) used in this segment of an algorithm:

```

t := 1
for i = n to n2
  t := t + 2it

```

48. Give a big- O estimate for the number of operations (where an operation is an addition or a multiplication) used in this segment of an algorithm:

```

t := 0
for i = 1 to n
  for j = 1 to n
    t := (it + jt + 1)2

```

In 49–52 assume that the number of multiplications of entries used to multiply a $p \times q$ and a $q \times r$ matrix is pqr .

49. What is the most efficient way to multiply the matrices $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3$ of sizes $20 \times 5, 5 \times 50, 50 \times 5$?
50. What is the most efficient way to multiply the matrices $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3$ of sizes $10 \times 50, 50 \times 10, 10 \times 40$?
51. What is the best order to form the product \mathbf{ABC} if \mathbf{A}, \mathbf{B} and \mathbf{C} are matrices with dimensions $2 \times 5, 5 \times 7$ and 7×3 , respectively?
52. What is the best order to form the product \mathbf{ABC} if \mathbf{A}, \mathbf{B} and \mathbf{C} are matrices with dimensions $8 \times 3, 3 \times 6$ and 6×12 , respectively?

Answers for Chapter 3

1. **procedure** *greaterthanfive*(a_1, \dots, a_n : integers)
answer := 0
for $i := 1$ **to** n
 if $a_i > 5$ **then** *answer* := *answer* + 1
return *answer*
2. **procedure** *secondlargest*(a_1, \dots, a_n : integers)
largest := a_1
secondlargest := a_2
if $a_2 > a_1$ **then**
 secondlargest := a_1
 largest := a_2
if $n = 2$ **then**
 return *secondlargest*
for $i := 3$ **to** n
if $a_i > largest$ **then**
 secondlargest := *largest*
 largest := a_i
if ($a_i > secondlargest$ **and** $a_i \leq largest$) **then**
 secondlargest := a_i
return *secondlargest*
3. **procedure** *lasteven*(a_1, \dots, a_n : integers)
location := 0
for $i := 1$ **to** n
if $2 \mid a_i$ **then** *location* := i
return *location*
4. **procedure** *avgmaxmin*(a_1, \dots, a_n : integers)
max := a_1
min := a_1
for $i := 2$ **to** n
 if $a_i > max$ **then** *max* := a_i
 if $a_i < min$ **then** *min* := a_i
return (*max* + *min*)/2
5. **procedure** *secondmax*(a_1, a_2, \dots, a_n : integers)
for $i = 2$ **to** n
 if $a_1 < a_i$ **then** exchange a_1 and a_i
secondmax := a_2
for $j = 3$ **to** n
 if *secondmax* < a_j **then** *secondmax* := a_j
return *secondmax* {*secondmax* is the second largest element}
6. **procedure** *largestproduct*(a_1, a_2, \dots, a_n, N : real numbers)
largestproduct := $-\infty$
for $i = 2$ **to** n
 for $j = 1$ **to** $i - 1$
 if $a_i \cdot a_j < N$ **then**
 if $a_i \cdot a_j > largestproduct$ **then** *largestproduct* := $a_i \cdot a_j$
return *largestproduct* {*largestproduct* is the largest product of two numbers in the list that is less than N , or $-\infty$ if all products are greater than or equal to N }
7. To search for x in an ordered list a_1, \dots, a_n , find the “midpoint” of the list and choose the appropriate half of the list. Continue until the list consists of one element. Either this element is x , or else x is not in the list.
8. The consecutive choices of sublists of the original list are: 15 21 25 31, 25 31, and 25. Since $27 \neq 25$, the integer 25 is not in the list.

9. The greedy algorithm first chooses a 12-foot-long board, and then three one-foot-long boards. This requires four boards. But only three boards are needed: each five feet long.
10. True. Note that each denomination divides the next largest one.
11. False. The algorithm gives change of 25 using 20, 1, 1, 1, 1, 1 (a total of six coins), but it can be done using 8, 8, 8, 1 (a total of only four coins).
12. $1^2 + 2^2 + \cdots + n^2 \leq n^2 + n^2 + \cdots + n^2 = n \cdot n^2 = n^3$.
13. $\left| \frac{3n - 8 - 4n^3}{2n - 1} \right| \leq \left| \frac{3n^3 + 8n^3 + 4n^3}{2n - n} \right| = \left| \frac{15n^3}{n} \right| = 15n^2$ if $n \geq 1$.
14. $1^3 + 2^3 + \cdots + n^3 \leq n^3 + n^3 + \cdots + n^3 = n \cdot n^3 = n^4$.
15. $\left| \frac{6n + 4n^5 - 4}{7n^2 - 3} \right| \leq \left| \frac{6n^5 + 4n^5}{7n^2 - n^2} \right| = \left| \frac{10n^5}{6n^2} \right| = \frac{5}{3}|n^3|$, if $n \geq 2$.
16. $1 \cdot 2 + 2 \cdot 3 + \cdots + (n-1) \cdot n \leq (n-1) \cdot n + (n-1) \cdot n + \cdots + (n-1) \cdot n = (n-1)^2 \cdot n \leq n^3$.
17. $f(n) \leq 3n^2 + 8n^2 + 7n^2 = 18n^2$ if $n \geq 1$; therefore $C = 18$ and $k = 1$ can be used.
18. n^2 .
19. n^2 .
20. n .
21. n^4 .
22. n^2 .
23. n^4 .
24. $\sum_{j=1}^n (j^3 + j) \leq \sum_{j=1}^n (n^3 + n^3) = n \cdot 2n^3 = 2n^4$.
25. $\log_2(x^2 + 1)$ and $\log_2(x^3 + 1)$ are each $O(\log_2 x)$. Thus each term is $O(x \log_2 x)$, and hence so is the sum.
26. n^3 .
27. x^2 .
28. $5x^4 + 2x^3 - 1$ is $O(x^4)$ since $|5x^4 + 2x^3 - 1| \leq |5x^4 + 2x^4| \leq 7|x^4|$ (if $x \geq 1$). Also, x^4 is $O(5x^4 + 2x^3 - 1)$ since $|x^4| \leq |5x^4 + x^3| \leq |5x^4 + 2x^3 - 1|$ (if $x \geq 1$).
29. $\frac{x^3 + 7x^2 + 3}{2x + 1}$ is $O(x^2)$ since $\frac{x^3 + 7x^2 + 3}{2x + 1} \leq \frac{x^3 + 7x^3 + 3x^3}{2x} = \frac{11x^3}{2x} = \frac{11}{2}x^2$ (if $x \geq 1$). Also, x^2 is $O\left(\frac{x^3 + 7x^2 + 3}{2x + 1}\right)$ since $x^2 = \frac{x^3}{x} \leq \frac{x^3 + 7x}{2x} \leq \frac{x^3 + 7x + 3}{2x + 1} \leq \frac{x^3 + 7x^2 + 3}{2x + 1}$ (if $x \geq 1$).
30. $x^3 + 7x + 2 \geq 1 \cdot x^3$ (if $x \geq 1$).
31. $\log(n!)$, $\log(n^n)$, $n^{3/2}$, $(n^{100})^n$
32. 10000 , $\log n^4$, $n^2 \log n$, $n^3 + 88n^2 + 3$, $n \cdot 2^n$, 3^n ,
33. $\log \log n$, $\log n^2$, $\log(n^2 + 1)$, $\log 2^n$, $n \log n$,
34. $(100n^3 + n^2, n^2 + n^3)$, $(3n^3 + 2^n, n^2 + 2^n)$.
35. The first algorithm uses fewer operations as n grows.
36. $\log_2 n$.
37. n .
38. $n!$.
39. 2^n .
40. n^2 .
41. $\log_2 n$.
42. n .
43. n .
44. n^3 .

45. 1.
46. n .
47. $O(n^2)$
48. $O(n^2)$
49. $\mathbf{A}_1(\mathbf{A}_2\mathbf{A}_3)$, 1750 multiplications.
50. $(\mathbf{A}_1\mathbf{A}_2)\mathbf{A}_3$, 9000 multiplications.
51. $(\mathbf{AB})\mathbf{C}$ uses $2 \cdot 5 \cdot 7 + 2 \cdot 7 \cdot 3 = 112$ multiplications, fewer than $\mathbf{A}(\mathbf{BC})$, which uses $5 \cdot 7 \cdot 3 + 2 \cdot 5 \cdot 3 = 135$.
52. $(\mathbf{AB})\mathbf{C}$ uses $8 \cdot 3 \cdot 6 + 8 \cdot 6 \cdot 12 = 720$ multiplications, more than $\mathbf{A}(\mathbf{BC})$, which uses $3 \cdot 6 \cdot 12 + 8 \cdot 3 \cdot 12 = 504$.

Questions for Chapter 4

1. What does a 60-second stop watch read 82 seconds after it reads 27 seconds?
 2. What does a 60-second stop watch read 54 seconds before it reads 19 seconds?
- In 3–6 suppose that a and b are integers, $a \equiv 4 \pmod{7}$, and $b \equiv 6 \pmod{7}$. Find the integer c with $0 \leq c \leq 6$ such that
3. $c \equiv 3a \pmod{7}$
 4. $c \equiv 5b \pmod{7}$
 5. $c \equiv 2a + 4b \pmod{7}$
 6. $c \equiv a^2 - b^2 \pmod{7}$
7. Prove or disprove: For all integers a, b, c, d , if $a|b$ and $c|d$, then $(a + c)|(b + d)$.
 8. Prove or disprove: For all integers a, b, c , if $a|b$ and $b|c$ then $a|c$.
 9. Prove or disprove: For all integers a, b, c , if $a|c$ and $b|c$, then $(a + b)|c$.
 10. Prove or disprove: For all integers a, b, c, d , if $a|b$ and $c|d$, then $(ac)|(b + d)$.
 11. Prove or disprove: For all integers a, b , if $a|b$ and $b|a$, then $a = b$.
 12. Prove or disprove: For all integers a, b, c , if $a|(b + c)$, then $a|b$ and $a|c$.
 13. Prove or disprove: For all integers a, b, c , if $a|bc$, then $a|b$ or $a|c$.
 14. Prove or disprove: For all integers a, b, c , if $a|c$ and $b|c$, then $ab|c^2$.
 15. Find the prime factorization of 1,024.
 16. Find the prime factorization of 1,025.
 17. Find the prime factorization of 510,510.
 18. Find the prime factorization of 8,827.
 19. Find the prime factorization of 45,617.
 20. Find the prime factorization of 111,111.
 21. List all positive integers less than 30 that are relatively prime to 20.
 22. Find $\gcd(20!, 12!)$ by directly finding the largest divisor of both numbers.

23. Find $\gcd(2^{89}, 2^{346})$ by directly finding the largest divisor of both numbers.
 24. Find $\text{lcm}(20!, 12!)$ by directly finding the smallest positive multiple of both numbers.
 25. Find $\text{lcm}(2^{89}, 2^{346})$ by directly finding the smallest positive multiple of both numbers.
 26. Suppose that the lcm of two numbers is 400 and their gcd is 10. If one of the numbers is 50, find the other number.
 27. Applying the division algorithm with $a = -41$ and $d = 6$ yields what value of r ?
 28. Find $18 \bmod 7$.
 29. Find $-88 \bmod 13$.
 30. Find $289 \bmod 17$.
 31. Find the hexadecimal expansion of $(ABC)_{16} + (2F5)_{16}$.
 32. Prove or disprove: A positive integer congruent to 1 modulo 4 cannot have a prime factor congruent to 3 modulo 4.
 33. Find $50! \bmod 50$.
 34. Find $50! \bmod 49!$.
 35. Prove or disprove: The sum of two primes is a prime.
 36. Prove or disprove: If p and q are primes (> 2), then $p + q$ is composite.
 37. Prove or disprove: There exist two consecutive primes, each greater than 2.
 38. Prove or disprove: The sum of two irrational numbers is irrational.
 39. Prove or disprove: If a and b are rational numbers (not equal to zero), then a^b is rational.
 40. Prove or disprove: If $f(n) = n^2 - n + 17$, then $f(n)$ is prime for all positive integers n .
 41. Prove or disprove: If p and q are primes (> 2), then $pq + 1$ is never prime.
 42. Find three integers m such that $13 \equiv 7 \pmod{m}$.
 43. Find the smallest integer $a > 1$ such that $a + 1 \equiv 2a \pmod{11}$.
 44. Find four integers b (two negative and two positive) such that $7 \equiv b \pmod{4}$.
 45. Find an integer a such that $a \equiv 3a \pmod{7}$.
 46. Find integers a and b such that $a + b \equiv a - b \pmod{5}$.
 47. Find $a \text{ div } m$ and $a \bmod m$ when $a = 76$, $m = 52$.
 48. Find $a \text{ div } m$ and $a \bmod m$ when $a = -33$, $m = 67$.
 49. Find $a \text{ div } m$ and $a \bmod m$ when $a = 511$, $m = 113$.
 50. Find the integer a such that $a \equiv 71 \pmod{47}$ and $-46 \leq a \leq 0$.
 51. Find the integer a such that $a \equiv 89 \pmod{19}$ and $-9 \leq a \leq 9$.
 52. Find the integer a such that $a \equiv 71 \pmod{41}$ and $160 \leq a \leq 200$.
- In 53–56 find each of these values
53. $(123 \bmod 19 + 342 \bmod 19) \bmod 19$.
 54. $(123 \bmod 19 \cdot 342 \bmod 19) \bmod 19$.
 55. $(12^2 \bmod 17)^3 \bmod 11$.

56. $(5^4 \bmod 7)^3 \bmod 13$.

57. Show that if a , b , k and m are integers such that $k \geq 1$, $m \geq 2$, and $a \equiv b \pmod{m}$, then $ka \equiv kb \pmod{m}$.

In questions 58–64 determine whether each of the following “theorems” is true or false. Assume that a , b , c , d , and m are integers with $m > 1$.

58. If $a \equiv b \pmod{m}$, and $a \equiv c \pmod{m}$, then $a \equiv b + c \pmod{m}$.

59. If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then $ac \equiv b + d \pmod{m}$.

60. If $a \equiv b \pmod{m}$, then $2a \equiv 2b \pmod{m}$.

61. If $a \equiv b \pmod{m}$, then $2a \equiv 2b \pmod{2m}$.

62. If $a \equiv b \pmod{m}$, then $a \equiv b \pmod{2m}$.

63. If $a \equiv b \pmod{2m}$, then $a \equiv b \pmod{m}$.

64. If $a \equiv b \pmod{m^2}$, then $a \equiv b \pmod{m}$.

65. Either find an integer x such that $x \equiv 2 \pmod{6}$ and $x \equiv 3 \pmod{9}$ are both true, or else prove that there is no such integer.

66. What sequence of pseudorandom numbers is generated using the pure multiplicative generator $x_{n+1} = 3x_n \bmod 11$ with seed $x_0 = 2$?

67. Explain in words the difference between $a|b$ and $\frac{b}{a}$.

68. Prove or disprove: if p and q are prime numbers, then $pq + 1$ is prime.

69. (a) Find two positive integers, each with exactly three positive integer factors greater than 1.

(b) Prove that there are an infinite number of positive integers, each with exactly three positive integer factors greater than 1.

70. Convert $(204)_{10}$ to base 2.

71. Convert $(1\ 1101)_2$ to base 16.

72. Convert $(1\ 1101)_2$ to base 10.

73. Convert $(2AC)_{16}$ to base 10.

74. Convert $(10,000)_{10}$ to base 2.

75. Convert $(8091)_{10}$ to base 2.

76. Convert $(BC1)_{16}$ to base 2.

77. Convert $(100\ 1100\ 0011)_2$ to base 16.

78. Convert $(271)_8$ to base 2.

79. Convert $(6253)_8$ to base 2.

80. Convert $(101011)_2$ to base 8.

81. Convert $(11010111100)_2$ to base 8.

In 82–83 find the sum and product of each of these pairs of numbers. Express your answer as a binary expansion.

82. $(101011)_2$, $(1101011)_2$

83. $(11010111100)_2$, $(11101110111)_2$

In 84–85 find the sum and product of each of these pairs of numbers. Express your answer as a base 3

expansion.

84. $(202)_3, (122)_3$

85. $(21202)_3, (12212)_3$

In 86–87 find the sum and product of each of these pairs of numbers. Express your answer as an octal expansion.

86. $(371)_8, (624)_8$

87. $(4274)_8, (5366)_8$

In 88–89 find the sum and product of each of these pairs of numbers. Express your answer as a hexadecimal expansion.

88. $(2A)_{16}, (BF)_{16}$

89. $(E3A)_{16}, (B5F8)_{16}$

90. Take any three-digit integer, reverse its digits, and subtract. For example, $742 - 247 = 495$. The difference is divisible by 9. Prove that this must happen for all three-digit numbers abc .

91. Prove or disprove that $30!$ ends in exactly seven 0's.

92. Here is a sample proof that contains an error. Explain why the proof is not correct.

Theorem: If $a|b$ and $b|c$, then $a|c$.

Proof: Since $a|b$, $b = ak$.

Since $b|c$, $c = bk$.

Therefore $c = bk = (ak)k = ak^2$.

Therefore $a|c$.

93. Prove: if n is an integer that is not a multiple of 3, then $n^2 \equiv 1 \pmod{3}$.

94. Prove: if n is an integer that is not a multiple of 4, then $n^2 \equiv 0 \pmod{4}$ or $n^2 \equiv 1 \pmod{4}$.

95. Use the Euclidean algorithm to find $\gcd(44, 52)$.

96. Use the Euclidean algorithm to find $\gcd(144, 233)$.

97. Use the Euclidean algorithm to find $\gcd(203, 101)$.

98. Use the Euclidean algorithm to find $\gcd(300, 700)$.

99. Use the Euclidean algorithm to find $\gcd(34, 21)$.

100. Use the Euclidean Algorithm to find $\gcd(900, 140)$.

101. Use the Euclidean Algorithm to find $\gcd(580, 50)$.

102. Use the Euclidean Algorithm to find $\gcd(390, 72)$.

103. Use the Euclidean Algorithm to find $\gcd(400, 0)$.

104. Use the Euclidean Algorithm to find $\gcd(128, 729)$.

105. Find the two's complement of 12.

106. Find the two's complement of -13 .

107. Find the two's complement of 9.

108. Given that $\gcd(620, 140) = 20$, write 20 as a linear combination of 620 and 140.

109. Given that $\gcd(662, 414) = 2$, write 2 as a linear combination of 662 and 414.

110. Express $\gcd(84, 18)$ as a linear combination of 18 and 84.

111. Express $\gcd(450, 120)$ as a linear combination of 120 and 450.

112. Find an inverse of 5 modulo 12.
113. Find an inverse of 17 modulo 19.
114. Solve the linear congruence $2x \equiv 5 \pmod{9}$.
115. Solve the linear congruence $5x \equiv 3 \pmod{11}$.
116. Find an inverse of 6 modulo 7.
117. Find an inverse of 5 modulo 17.
118. Find an inverse of 2 modulo 31.
119. Solve the linear congruence $15x \equiv 31 \pmod{47}$ given that the inverse of 15 modulo 47 is 22.
120. Solve the linear congruence $54x \equiv 12 \pmod{73}$ given that the inverse of 54 modulo 73 is 23.
121. Solve the linear congruence $31x \equiv 57 \pmod{61}$.
122. Use Fermat's little theorem to find $9^{45} \pmod{23}$.
123. Use Fermat's little theorem to find $25^{1202} \pmod{61}$.
124. Show that 7 is a primitive root of 13.
125. Find the discrete logarithms of 5 and 8 to the base 7 modulo 13.
126. Find the first five terms of the sequence of four-digit pseudorandom numbers generated by the middle square method starting with 1357.
127. Find the first five terms of the sequence of four-digit pseudorandom numbers generated by the middle square method starting with 9361.
128. Find the sequence of pseudorandom numbers generated by the power generator $x_{n+1} = x_n^2 \pmod{17}$, and seed $x_0 = 5$.
129. Find the sequence of pseudorandom numbers generated by the power generator $x_{n+1} = x_n^3 \pmod{23}$, and seed $x_0 = 3$.
- The numbers in question 130–133 refer to an 8-digit student id at a large university. The eighth digit is a check digit equal to the sum of the first seven digits modulo 7.
130. Find the check digit of the student id starting with 2365 415.
131. Find the check digit of the student id starting with 3179 822.
132. Suppose the first digit of the student id X123 4566 is illegible (indicated by X). Can you tell what the first digit has to be?
133. Suppose the first digit of the student id X923 4562 is illegible (indicated by X). Can you tell what the first digit has to be?
134. Encrypt the message NEED HELP by translating the letters into numbers (A=0, B=1, ..., Z=25), applying the encryption function $f(p) = (p + 3) \pmod{26}$, and then translating the numbers back into letters.
135. Encrypt the message NEED HELP by translating the letters into numbers (A=0, B=1, ..., Z=25), applying the encryption function $f(p) = (3p + 7) \pmod{26}$, and then translating the numbers back into letters.
136. Suppose that a computer has only the memory locations 0, 1, 2, ..., 19. Use the hashing function h where $h(x) = (x + 5) \pmod{20}$ to determine the memory locations in which 57, 32, and 97 are stored.
137. A message has been encrypted using the function $f(x) = (x + 5) \pmod{26}$. If the message in coded form is JCFHY, decode the message.
138. Explain why $f(x) = (2x + 3) \pmod{26}$ would not be a good coding function.

139. Encode the message “stop at noon” using the function $f(x) = (x + 6) \bmod 26$.
140. Encrypt the message “just testing” using the function $f(x) = (5x + 3) \bmod 26$.
141. Encrypt the message “meet me at noon” using the function $f(x) = (9x + 1) \bmod 26$.
142. Decrypt the message “AHFXVHFBGZ” that was encrypted using the shift cipher $f(x) = (x + 19) \bmod 26$.
143. What is the decryption function for an affine cipher if the encryption function is $f(x) = (3x + 7) \bmod 26$?
144. Encrypt the message WATCH OUT using blocks of four letters and the transposition cipher based on the permutation of $\{1, 2, 3, 4\}$ with $\sigma(1) = 3$, $\sigma(2) = 4$, $\sigma(3) = 2$, and $\sigma(4) = 1$.
145. Encrypt the message CANCEL THE ORDER using blocks of seven letters and the transposition cipher based on the permutation of $\{1, 2, 3, 4, 5, 6, 7\}$ with $\sigma(1) = 5$, $\sigma(2) = 3$, $\sigma(3) = 6$, $\sigma(4) = 1$, $\sigma(5) = 7$, $\sigma(6) = 2$, and $\sigma(7) = 4$.
146. Decrypt the message EARLYL which is the ciphertext produced by encrypting a plaintext message using the transposition cipher with blocks of three letters and the permutation σ of $\{1, 2, 3\}$ defined by $\sigma(1) = 3$, $\sigma(2) = 1$, and $\sigma(3) = 2$.
147. Use the Vigenère cipher with key LOCK to encrypt the message NEXT FALL.
148. Use the Vigenère cipher with key NOW to encrypt the message SUMMER.
149. The cipher text LTDTLWW was produced by encrypting a plaintext message using the Vigenère cipher with the key TEST. What is the plaintext message?
150. Encrypt the message KING using the RSA system with $n = 43 \cdot 61$ and $e = 13$, translating each letter into integers (A = 00, B=01, ...) and grouping together pairs of integers.
151. Encrypt the message BALL using the RSA system with $n = 37 \cdot 73$ and $e = 7$, translating each letter into integers (A = 00, B=01, ...) and grouping together pairs of integers.
152. What is the shared key if Alice and Bob use the Diffie-Hellman key exchange protocol with the prime $p = 67$, the primitive root $a = 7$ of $p = 67$, with Alice choosing the secret integer $k_1 = 12$ and Bob choosing the secret integer $k_2 = 25$?
153. What is the shared key if Alice and Bob use the Diffie-Hellman key exchange protocol with the prime $p = 431$, the primitive root $a = 79$ of $p = 431$, with Alice choosing the secret integer $k_1 = 236$ and Bob choosing the secret integer $k_2 = 334$?
154. Alice has the public key $(n, e) = (2623, 13)$ with corresponding private key $d = 1357$, and she wants to send the message LAST CALL to her friends so that they know she sent it. What should she send to her friends, assuming she signs the message using the RSA cryptosystem?

Answers for Chapter 4

1. 49 seconds
2. 25 seconds
3. 5
4. 2
5. 4
6. 1
7. False: $a = b = c = 1, d = 2$.
8. True: If $b = ak$ and $c = bl$, then $c = a(kl)$, so $a \mid c$.

9. False: $a = b = c = 1$.
10. False: $a = b = 2, c = d = 1$.
11. False: $a = 1, b = -1$.
12. False: $a = 2, b = c = 3$.
13. False: $a = 4, b = 2, c = 6$.
14. True: If $c = ak$ and $c = bl$, then $c^2 = ab(kl)$, so $ab \mid c^2$.
15. 2^{10} .
16. $5^2 \cdot 41$.
17. $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17$.
18. $7 \cdot 13 \cdot 97$.
19. $11^2 \cdot 13 \cdot 29$.
20. $3 \cdot 7 \cdot 11 \cdot 13 \cdot 37$.
21. 1, 3, 7, 9, 11, 13, 17, 19, 21, 23, 27, 29.
22. $12!$.
23. 2^{89} .
24. $20!$.
25. 2^{346} .
26. 80.
27. 1.
28. 4.
29. 3.
30. 0.
31. $(DB1)_{16}$.
32. False: $9 = 4 \cdot 2 + 1 = 3 \cdot 3$.
33. 0.
34. 0.
35. False; $3 + 5$ is not prime.
36. $p + q$ is even, hence composite.
37. False; one of any two consecutive integers is even, hence not prime.
38. False; $\sqrt{2} + (-\sqrt{2}) = 0$.
39. False; $(1/2)^{1/2} = \sqrt{2}/2$, which is not rational.
40. False, $f(17)$ is divisible by 17.
41. $pq + 1$ is an even number, hence not prime.
42. 2, 3, 6.
43. 12.
44. 3, 7, 11, 15, \dots , -1, -5, -9, \dots
45. 0, ± 7 , ± 14 , \dots
46. $b = 0, \pm 5, \pm 10, \pm 15, \dots$; a any integer.
47. 1, 24
48. -1, 34
49. 4, 59
50. -23

51. -6
52. 194
53. 9
54. 0
55. 6
56. 8
57. The hypothesis $a \equiv b \pmod{m}$ means that $m|(a - b)$. Therefore $m|(k \cdot (a - b))$, which means precisely that $ka \equiv kb \pmod{m}$.
58. False.
59. False.
60. True.
61. True.
62. False.
63. True.
64. True.
65. There is no such x ; if there were, then there would be integers k and l such that $x - 2 = 6k$ and $x - 3 = 9l$. Hence $1 = 6k - 9l = 3(2k - 3l)$, which is not possible.
66. The sequence 2, 6, 7, 10, 8 repeats.
67. $a | b$ is a statement; $\frac{b}{a}$ represents a number.
68. False: $p = q = 3$.
69. (a) 8, 27. (b) Any integer of the form p^3 where p is prime.
70. 1100 1100.
71. 1D.
72. 29.
73. 684.
74. 10 0111 0001 0000.
75. 1 1111 1001 1011.
76. 1011 1100 0001.
77. 4C3.
78. $(1011 1001)_2$
79. $(1100 1010 1011)_2$
80. $(53)_8$
81. $(3274)_8$
82. $(1001 0110)_2$, $(1 0001 1111 1001)_2$
83. $(1110 0011 0011)_2$, $(11 0010 0100 0101 0110 0100)_2$
84. $(1101)_3$, $(110121)_3$
85. $(111121)_3$, $(1200022001)_3$
86. $(1215)_8$, $(304364)_8$
87. $(11662)_8$, $(27736250)_8$
88. $(E9)_{16}$, $(1F56)_{16}$
89. $(C432)_{16}$, $(A1CCA30)_{16}$
90. $abc - cba = 100a + 10b + c - (100c + 10b + a) = 99a - 99c = 9(11a - 11c)$. Therefore $9 | abc - cba$.

91. True. When the factors 5, 10, 15, 20, and 30 are multiplied by the factor 2, each contributes one zero; when the factor 25 is multiplied by two factors 2, it contributes two zeros.
92. The proof is not correct since there is no guarantee that the multiple k will be the same in both cases.
93. Proof by cases. Suppose n is not a multiple of 3. Then $n = 3k + 1$ or $n = 3k + 2$ for some integer k .
Case 1, $n = 3k + 1$: therefore $n^2 = (3k + 1)^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1$, and hence $n^2 \equiv 1 \pmod{3}$.
Case 2, $n = 3k + 2$: therefore $n^2 = (3k + 2)^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k + 1) + 1$, and hence $n^2 \equiv 1 \pmod{3}$.
94. Proof by cases. Suppose n is not a multiple of 4. Then there is an integer k such that $n = 4k + 1$, $n = 4k + 2$, or $n = 4k + 3$.
Case 1, $n = 4k + 1$: therefore $n^2 = (4k + 1)^2 = 16k^2 + 8k + 1 = 4(4k^2 + 2k) + 1$, and hence $n^2 \equiv 1 \pmod{4}$.
Case 2, $n = 4k + 2$: therefore $n^2 = (4k + 2)^2 = 16k^2 + 16k + 4 = 4(4k^2 + 4k + 1)$, and hence $n^2 \equiv 0 \pmod{4}$.
Case 3, $n = 4k + 3$: therefore $n^2 = (4k + 3)^2 = 16k^2 + 24k + 9 = 4(4k^2 + 6k + 2) + 1$, and hence $n^2 \equiv 1 \pmod{4}$.
95. 4.
96. 1.
97. 1.
98. 100.
99. 1.
100. 20.
101. 10.
102. 6.
103. 400.
104. 1.
105. 0 1100.
106. 1 0011.
107. 0 1001.
108. $620 \cdot (-2) + 140 \cdot 9$.
109. $662 \cdot (-5) + 414 \cdot 8$.
110. $18 \cdot (-9) + 84 \cdot 2$.
111. $120 \cdot 4 + 450 \cdot (-1)$.
112. 5.
113. 9.
114. $7 + 9k$.
115. $5 + 11k$.
116. 6
117. 7
118. 16
119. 24
120. 57
121. 53
122. 9
123. 15
124. The powers of 7 modulo 13 are 7, 10, 5, 9, 11, 12, 6, 3, 8, 4, 2, 1.
125. 3, 9
126. 1357, 8414, 7953, 2502, 2600
127. 9361, 6283, 4760, 6576, 2437

128. 8, 13, 16, 1, 1, 1, ...
 129. 4, 18, 13, 12, 3, 4, ...
 130. 5
 131. 4
 132. 6
 133. 1 or 8
 134. Encrypted form: QHHG KHOS.
 135. Encrypted form: UTTQ CTOA.
 136. 2, 17, 3.
 137. EXACT.
 138. f is not 1-1 ($f(0) = f(13)$), and hence f^{-1} is not a function.
 139. YZUV GZ TUUT.
 140. WZPU UXPURQH
 141. FLLQ FL BQ OXXO
 142. HOMECOMING
 143. $f(x) = 9x + 15$
 144. CTWA TUHO
 145. CLATCNE REERHOD
 146. REA LLY
 147. YSZD QONV
 148. FIIZSN
 149. SPLASHED
 150. 1740 2314
 151. 1506 0075
 152. $(7^{12})^{25} \bmod 67 = 59$
 153. $(79^{236})^{334} \bmod 431 = 353$
 154. 1519 2124 2227 1111

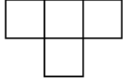
Questions for Chapter 5

1. Suppose you wish to prove that the following is true for all positive integers n by using the Principle of Mathematical Induction: $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

(a) Write $P(1)$	(b) Write $P(72)$
(c) Write $P(73)$	(d) Use $P(72)$ to prove $P(73)$
(e) Write $P(k)$	(f) Write $P(k + 1)$
(g) Use the Principle of Mathematical Induction to prove that $P(n)$ is true for all positive integers n	

2. Suppose you wish to use the Principle of Mathematical Induction to prove that $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + n \cdot n! = (n + 1)! - 1$ for all $n \geq 1$.

(a) Write $P(1)$	(b) Write $P(5)$
(c) Write $P(k)$	(d) Write $P(k + 1)$
(e) Use the Principle of Mathematical Induction to prove that $P(n)$ is true for all $n \geq 1$	

3. Use the Principle of Mathematical Induction to prove that $1 - 2 + 2^2 - 2^3 + \cdots + (-1)^n 2^n = \frac{2^{n+1}(-1)^n + 1}{3}$ for all positive integers n .
 4. Use the Principle of Mathematical Induction to prove that $1 + 2^n \leq 3^n$ for all $n \geq 1$.
 5. Use the Principle of Mathematical Induction to prove that $n^3 > n^2 + 3$ for all $n \geq 2$.
 6. Use the Principle of Mathematical Induction to prove that $2|(n^2 + n)$ for all $n \geq 0$.
 7. Use the Principle of Mathematical Induction to prove that $1 + 3 + 9 + 27 + \cdots + 3^n = \frac{3^{n+1} - 1}{2}$ for all $n \geq 0$.
 8. Use the Principle of Mathematical Induction to prove that $1 + 4 + 7 + 10 + \cdots + (3n - 2) = \frac{n(3n - 1)}{2}$ for all $n \geq 1$.
 9. Use the Principle of Mathematical Induction to prove that $2|(n^2 + 3n)$ for all $n \geq 1$.
 10. Use the Principle of Mathematical Induction to prove that $2n + 3 \leq 2^n$ for all $n \geq 4$.
 11. Use the Principle of Mathematical Induction to prove that $3|(n^3 + 3n^2 + 2n)$ for all $n \geq 1$.
 12. Use the Principle of Mathematical Induction to prove that any integer amount of postage from 18 cents on up can be made from an infinite supply of 4-cent and 7-cent stamps.
 13. Suppose that the only paper money consists of 3-dollar bills and 10-dollar bills. Show that any dollar amount greater than 17 dollars could be made from a combination of these bills.
 14. Use mathematical induction to prove that every integer amount of postage of six cents or more can be formed using 3-cent and 4-cent stamps.
 15. Prove that $\sum_{j=n}^{2n-1} (2j + 1) = 3n^2$ for all positive integers n .
 16. Use mathematical induction to show that n lines in the plane passing through the same point divide the plane into $2n$ regions.
 17. Let $a_1 = 2$, $a_2 = 9$, and $a_n = 2a_{n-1} + 3a_{n-2}$ for $n \geq 3$. Show that $a_n \leq 3^n$ for all positive integers n .
 18. Floor borders one foot wide and of varying lengths are to be covered with nonoverlapping tiles that are available in two sizes: $1' \times 3'$ and $1' \times 5'$ sizes. Assuming that the supply of each size is infinite, prove that every $1' \times n'$ border ($n > 7$) can be covered with these tiles.
 19. A *T-omino* is a tile pictured at the right. Prove that every $2^n \times 2^n$ ($n > 1$) chessboard can be tiled with *T-ominoes*.
 
 20. Use the Principle of Mathematical Induction to prove that $4|(9^n - 5^n)$ for all $n \geq 0$.
 21. Use the Principle of Mathematical Induction to prove that $5|(7^n - 2^n)$ for all $n \geq 0$.
 22. Prove that the distributive law $A_1 \cap (A_2 \cup \cdots \cup A_n) = (A_1 \cap A_2) \cup \cdots \cup (A_1 \cap A_n)$ is true for all $n \geq 3$.
 23. Prove that $\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \cdots + \frac{n}{2^n} = \frac{2^{n+1} - 2 - n}{2^n}$ for all $n \geq 1$.
 24. Find the error in the following proof of this “theorem”:

“Theorem: Every positive integer equals the next largest positive integer.”

“Proof: Let $P(n)$ be the proposition ‘ $n = n + 1$ ’. To show that $P(k) \rightarrow P(k + 1)$, assume that $P(k)$ is true for some k , so that $k = k + 1$. Add 1 to both sides of this equation to obtain $k + 1 = k + 2$, which is $P(k + 1)$. Therefore $P(k) \rightarrow P(k + 1)$ is true. Hence $P(n)$ is true for all positive integers n .”
- In questions 25–33 give a recursive definition with initial condition(s).
25. The function $f(n) = 2^n$, $n = 1, 2, 3, \dots$

26. The function $f(n) = n!$, $n = 0, 1, 2, \dots$
27. The function $f(n) = 5n + 2$, $n = 1, 2, 3, \dots$
28. The sequence $a_1 = 16$, $a_2 = 13$, $a_3 = 10$, $a_4 = 7$, \dots
29. The Fibonacci numbers 1, 1, 2, 3, 5, 8, 13, \dots
30. The set $\{0, 3, 6, 9, \dots\}$.
31. The set $\{1, 5, 9, 13, 17, \dots\}$.
32. The set $\{1, 1/3, 1/9, 1/27, \dots\}$.
33. The set $\{\dots, -4, -2, 0, 2, 4, 6, \dots\}$.

In questions 34–39 give a recursive definition (with initial condition(s)) of $\{a_n\}$ ($n = 1, 2, 3, \dots$).

34. $a_n = 2^n$.
35. $a_n = 3n - 5$.
36. $a_n = (n + 1)/3$.
37. $a_n = \sqrt{2}$.
38. $a_n = 2^{1/2^n}$.
39. $a_n = n^2 + n$.

In questions 40–44 give a recursive definition with initial condition(s) of the set S .

40. $\{3, 7, 11, 15, 19, 23, \dots\}$.
41. All positive integer multiples of 5.
42. $\{\dots, -5, -3, -1, 1, 3, 5, \dots\}$.
43. $\{0.1, 0.01, 0.001, 0.0001\}$.
44. The set of strings 1, 111, 11111, 1111111, \dots
45. Find $f(2)$ and $f(3)$ if $f(n) = 2f(n - 1) + 6$, $f(0) = 3$.
46. Find $f(2)$ and $f(3)$ if $f(n) = f(n - 1) \cdot f(n - 2) + 1$, $f(0) = 1$, $f(1) = 4$.
47. Find $f(2)$ and $f(3)$ if $f(n) = f(n - 1)/f(n - 2)$, $f(0) = 2$, $f(1) = 5$.
48. Suppose $\{a_n\}$ is defined recursively by $a_n = a_{n-1}^2 - 1$ and that $a_0 = 2$. Find a_3 and a_4 .
49. Give a recursive algorithm for computing na , where n is a positive integer and a is a real number.
50. Describe a recursive algorithm for computing 3^{2^n} where n is a nonnegative integer.

51. Verify that the program segment

```
a := 2
b := a + c
```

is correct with respect to the initial assertion $c = 3$ and the final assertion $b = 5$.

52. Consider the following program segment:

```
i := 1
total := 1
while i < n
  i := i + 1
  total := total + i
```

Let p be the proposition “ $total = \frac{i(i+1)}{2}$ and $i \leq n$.” Use mathematical induction to prove that p is a loop

invariant.

53. Verify that the following program segment is correct with respect to the initial assertion **T** and the final assertion $(x \leq y \wedge \max = y) \vee (x > y \wedge \max = x)$:

```

if  $x \leq y$  then
   $\max := y$ 
else
   $\max := x$ 

```

Answers for Chapter 5

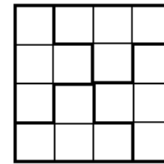
- $1 = 1^2$.
 - $1 + 3 + 5 + \cdots + 143 = 72^2$.
 - $1 + 3 + 5 + \cdots + 145 = 73^2$.
 - $1 + 3 + 5 + \cdots + 145 = (1 + 3 + 5 + \cdots + 143) + 145 = 72^2 + 145 = 72^2 + 2 \cdot 72 + 1 = (72 + 1)^2 = 73^2$.
 - $1 + 3 + \cdots + (2k - 1) = k^2$.
 - $1 + 3 + \cdots + (2k + 1) = (k + 1)^2$.
 - $P(1)$ is true since $1 = 1^2$. $P(k) \rightarrow P(k + 1)$: $1 + 3 + \cdots + (2k + 1) = k^2 + (2k + 1) = (k + 1)^2$.
- $1 \cdot 1! = 2! - 1$.
 - $1 \cdot 1! + 2 \cdot 2! + \cdots + 5 \cdot 5! = 6! - 1$.
 - $1 \cdot 1! + 2 \cdot 2! + \cdots + k \cdot k! = (k + 1)! - 1$.
 - $1 \cdot 1! + 2 \cdot 2! + \cdots + (k + 1)(k + 1)! = (k + 2)! - 1$.
 - $P(1)$ is true since $1 \cdot 1! = 1$ and $2! - 1 = 1$. $P(k) \rightarrow P(k + 1)$: $1 \cdot 1! + 2 \cdot 2! + \cdots + (k + 1)(k + 1)! = (k + 1)! - 1 + (k + 1)(k + 1)! = (k + 1)![1 + (k + 1)] - 1 = (k + 1)!(k + 2) - 1 = (k + 2)! - 1$.
- $P(1)$: $1 - 2 = \frac{2^2(-1) + 1}{3}$, which is true since both sides are equal to -1 . $P(k) \rightarrow P(k + 1)$: $1 - 2 + 2^2 + \cdots + (-1)^{k+1}2^{k+1} = \frac{2^{k+1}(-1)^k + 1}{3} + (-1)^{k+1}2^{k+1} = \frac{2^{k+1}(-1)^k + 1 + 3(-1)^{k+1}2^{k+1}}{3}$
 $= \frac{2^{k+1}(-1)^k(1 + 3(-1)) + 1}{3} = \frac{2^{k+1}(-1)^k(-2) + 1}{3} = \frac{2^{k+2}(-1)^{k+1} + 1}{3}$.
- $P(1)$: $1 + 2^1 \leq 3^1$, which is true since both sides are equal to 3. $P(k) \rightarrow P(k + 1)$: $1 + 2^{k+1} = (1 + 2^k) + 2^k \leq 3^k + 2^k \leq 3^k + 3^k = 2 \cdot 3^k < 3 \cdot 3^k = 3^{k+1}$.
- $P(2)$: $2^3 > 2^2 + 3$ is true since $8 > 7$. $P(k) \rightarrow P(k + 1)$: $(k + 1)^2 + 3 = k^2 + 2k + 1 + 3 = (k^2 + 3) + 2k + 1 < k^3 + 2k + 1 \leq k^3 + 3k \leq k^3 + 3k^2 + 3k + 1 = (k + 1)^3$.
- $P(0)$: $2 \mid 0^2 + 0$, which is true since $2 \mid 0$. $P(k) \rightarrow P(k + 1)$: $(k + 1)^2 + (k + 1) = (k^2 + k) + 2(k + 1)$, which is divisible by 2 since $2 \mid k^2 + k$ and $2 \mid 2(k + 1)$.
- $P(0)$: $1 = \frac{3^1 - 1}{2}$, which is true since $1 = 1$. $P(k) \rightarrow P(k + 1)$: $1 + 3 + \cdots + 3^{k+1} = \frac{3^{k+1} - 1}{2} + 3^{k+1} = \frac{3^{k+1} - 1 + 2 \cdot 3^{k+1}}{2} = \frac{3^{k+2} - 1}{2}$.
- $P(1)$: $1 = \frac{1 \cdot 2}{2}$, which is true since $1 = 1$. $P(k) \rightarrow P(k + 1)$: $1 + 4 + \cdots + (3(k + 1) - 2) = \frac{k(3k - 1)}{2} + (3k + 1) = \frac{k(3k - 1) + 2(3k + 1)}{2} = \frac{3k^2 + 5k + 2}{2} = \frac{(3k + 2)(k + 1)}{2} = \frac{(k + 1)(3(k + 1) - 1)}{2}$.
- $P(1)$: $2 \mid 1^2 + 3 \cdot 1$, which is true since $2 \mid 4$. $P(k) \rightarrow P(k + 1)$: $(k + 1)^2 + 3(k + 1) = (k^2 + 3k) + 2(k + 2)$, which is divisible by 2 since $2 \mid k^2 + 3k$ and $2 \mid 2(k + 2)$.
- $P(4)$: $2 \cdot 4 + 3 \leq 2^4$, which is true since $11 \leq 16$. $P(k) \rightarrow P(k + 1)$: $2(k + 1) + 3 = (2k + 3) + 2 \leq 2^k + 2 \leq 2^k + 2^k = 2^{k+1}$.
- $P(1)$: $3 \mid 1^3 + 3 \cdot 1^2 + 2 \cdot 1$, which is true since $3 \mid 6$. $P(k) \rightarrow P(k + 1)$: $(k + 1)^3 + 3(k + 1)^2 + 2(k + 1) = (k^3 + 3k^2 + 2k) + 3(k^2 + 3k + 2)$, which is divisible by 3 since each of the two terms is divisible by 3.

12. $P(18)$: use one 4-cent stamp and two 7-cent stamps. $P(k) \rightarrow P(k+1)$: if a pile of stamps for k cents postage has a 7-cent stamp, replace one 7-cent stamp with two 4-cent stamps; if the pile contains only 4-cent stamps (there must be at least five of them), replace five 4-cent stamps with three 7-cent stamps.
13. $P(18)$: Eighteen dollars can be made using six 3-dollar bills. $P(k) \rightarrow P(k+1)$: Suppose that k dollars can be formed, for some $k \geq 18$. If at least two 10-dollar bills are used, replace them by seven 3-dollar bills to form $k+1$ dollars. Otherwise (that is, at most one 10-dollar bill is used), at least three 3-dollar bills are being used, and three of them can be replaced by one 10-dollar bill to form $k+1$ dollars.
14. $P(6)$: Six cents postage can be made from two 3-cent stamps. $P(k) \rightarrow P(k+1)$: either replace a 3-cent stamp by a 4-cent stamp or else (if there are only 4-cent stamps in the pile of stamps making k cents postage) replace two 4-cent stamps by three 3-cent stamps.

15. The basis case holds since $\sum_{j=1}^1 (2j+1) = 3 = 3 \cdot 1^2$. Now assume that $\sum_{j=k}^{2k-1} (2j+1) = 3k^2$ for some k . It follows

$$\text{that } \sum_{j=k+1}^{2(k+1)-1} (2j+1) = \sum_{j=k}^{2k-1} (2j+1) - (2k+1) + (4k+1) + (4k+3) = 3k^2 + 6k + 3 = 3(k+1)^2.$$

16. The basis step follows since one line divides the plane into 2 regions. Now assume that k lines passing through the same point divide the plane into $2k$ regions. Adding the $(k+1)$ st line splits exactly two of these regions into two parts each. Hence, the $k+1$ lines split the plane into $2k+2 = 2(k+1)$ regions.
17. Let $P(n)$ be the proposition that $a_n \leq 3^n$. The proof uses the Principle of Strong Induction. The basis step follows since $a_1 = 2 \leq 3 = 3^1$ and $a_2 = 9 \leq 9 = 3^2$. Now assume that $P(k)$ is true for all k such that $1 \leq k < n$ (where $n \geq 3$). Then $a_k \leq 3^k$ for $1 \leq k < n$. Hence $a_n = 2a_{n-1} + 3a_{n-2} \leq 2 \cdot 3^{n-1} + 3 \cdot 3^{n-2} = 2 \cdot 3^{n-1} + 3^{n-1} = 3 \cdot 3^{n-1} = 3^n$.
18. $P(8)$: use one of each type. $P(k) \rightarrow P(k+1)$: If a $1' \times 5'$ tile is used as part of the covering of a $1' \times k'$ strip, replace a $1' \times 5'$ tile with two $1' \times 3'$ tiles to cover a $1' \times (k+1)'$ strip. Otherwise, the tiles for the $1' \times k'$ strip must include three $1' \times 3'$ tiles; replace three of these with two $1' \times 5'$ tiles to cover a $1' \times (k+1)'$ strip.
19. $P(2)$: The figure at the right shows a tiling of a 4×4 board.
 $P(k) \rightarrow P(k+1)$: Divide the $2^{k+1} \times 2^{k+1}$ board into four quarters, each of which is a $2^k \times 2^k$ board. $P(k)$ guarantees that each of these four $2^k \times 2^k$ boards can be tiled. Put these four tiled boards together to obtain a tiling for the $2^{k+1} \times 2^{k+1}$ board.



20. $P(0)$: $4 \mid 1 - 1$ is true since $4 \mid 0$. $P(k) \rightarrow P(k+1)$: $9^{k+1} - 5^{k+1} = 9(9^k - 5^k) + 5^k(9 - 5)$. Each term is divisible by 4: $4 \mid 9^k - 5^k$ (by $P(k)$) and $4 \mid 9 - 5$.
21. $P(1)$: $5 \mid 7 - 2$ is true since $5 \mid 5$. $P(k) \rightarrow P(k+1)$: $7^{k+1} - 2^{k+1} = 7(7^k - 2^k) + 2^k(7 - 2)$. Each term is divisible by 5: $5 \mid 7^k - 2^k$ (by $P(k)$) and $5 \mid 7 - 2$.
22. The second form of mathematical induction is used. $P(3) \wedge \dots \wedge P(n) \rightarrow P(n+1)$: $A_1 \cap (A_2 \cup \dots \cup A_{n+1}) = A_1 \cap ((A_2 \cup \dots \cup A_n) \cup A_{n+1}) = [A_1 \cap (A_2 \cup \dots \cup A_n)] \cup (A_1 \cap A_{n+1}) = [(A_1 \cap A_2) \cup \dots \cup (A_1 \cap A_n)] \cup (A_1 \cap A_{n+1}) = (A_1 \cap A_2) \cup \dots \cup (A_1 \cap A_{n+1})$.
23. $P(1)$: $\frac{1}{2} = \frac{(2^2-2-1)}{2^1}$, which is true since the right side is equal to $1/2$. $P(k) \rightarrow P(k+1)$: $\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \dots + \frac{k+1}{2^{k+1}} = \frac{2^{k+1}-2-k}{2^k} + \frac{k+1}{2^{k+1}} = \frac{2^{k+2}-4-2k+k+1}{2^{k+1}} = \frac{2^{k+2}-3-k}{2^{k+1}} = \frac{2^{k+2}-2-(k+1)}{2^{k+1}}$.
24. No basis case has been shown.
25. $f(n) = 2f(n-1)$, $f(1) = 2$.
26. $f(n) = nf(n-1)$, $f(0) = 1$.
27. $f(n) = f(n-1) + 5$, $f(1) = 7$.
28. $a_n = a_{n-1} - 3$, $a_1 = 16$.
29. $a_n = a_{n-1} + a_{n-2}$, $a_1 = 1$, $a_2 = 1$.
30. $0 \in S$; $x \in S \rightarrow x + 3 \in S$.
31. $1 \in S$; $x \in S \rightarrow x + 4 \in S$.

32. $1 \in S; x \in S \rightarrow x/3 \in S$.
33. $0 \in S; x \in S \rightarrow x \pm 2 \in S$.
34. $a_n = 2a_{n-1}, a_1 = 2$.
35. $a_n = a_{n-1} + 3, a_1 = -2$.
36. $a_n = a_{n-1} + 1/3, a_1 = 2/3$.
37. $a_n = a_{n-1}, a_1 = \sqrt{2}$.
38. $a_n = \sqrt{a_{n-1}}, a_1 = \sqrt{2}$.
39. $a_n = a_{n-1} + 2n, a_1 = 2$.
40. $3 \in S; x \in S \rightarrow x + 4 \in S$.
41. $5 \in S; x \in S \rightarrow x + 5 \in S$.
42. $1 \in S; x \in S \rightarrow x \pm 2 \in S$.
43. $0.1 \in S; x \in S \rightarrow x/10 \in S$.
44. $1 \in S; x \in S \rightarrow x11 \in S$ (or $x \in S \rightarrow 100x + 11 \in S$).
45. $f(2) = 30, f(3) = 66$.
46. $f(2) = 5, f(3) = 21$.
47. $f(2) = 5/2, f(3) = 1/2$.
48. $a_3 = 63$ and $a_4 = 3,968$.
49. The following procedure computes na :
procedure *mult*(a : real number, n : positive integer)
if $n = 1$ **then** $mult(a, n) := a$
else $mult(a, n) := a + mult(a, n - 1)$.
50. The following procedure computes 3^{2^n} :
procedure *power*(n : nonnegative integer)
if $n = 0$ **then** $power(n) := 3$
else $power(n) := power(n - 1) \cdot power(n - 1)$.
51. Suppose $c = 3$. The program segment assigns 2 to a and then assigns $a + c = 2 + 3 = 5$ to b .
52. Before the loop is entered p is true since $total = \frac{1 \cdot 2}{2}$ and $i \leq n$. Suppose p is true and $i < n$ after an execution of the loop. Suppose that the **while** loop is executed again. The variable i is incremented by 1, and hence $i \leq n$. The variable $total$ was $\frac{(i-1)i}{2}$, which now becomes $\frac{(i-1)i}{2} + i = \frac{i(i+1)}{2}$. Hence p is a loop invariant.
53. If $x < y$ initially, max is set equal to y , so $(x \leq y \wedge max = y)$ is true. If $x = y$ initially, max is set equal to y , so $(x \leq y \wedge max = y)$ is again true. If $x > y$, max is set equal to x , so $(x > y \wedge max = x)$ is true.

Questions for Chapter 6

In questions 1–12 suppose that a “word” is any string of seven letters of the alphabet, with repeated letters allowed.

- How many words are there?
- How many words end with the letter T?
- How many words begin with R and end with T?
- How many words begin with A or B?
- How many words begin with A or end with B?

6. How many words begin with A or B and end with A or B?
7. How many words begin with A or B or end with A or B?
8. How many words begin with a vowel and end with a vowel?
9. How many words begin with a vowel or end with a vowel?
10. How many words begin with AAB in some order?
11. How many words have no vowels?
12. How many words have exactly one vowel?
13. Find the number of words of length eight of distinct letters of the alphabet so that the words do not have both *A* and *B* in them.

In questions 14–18 consider all bit strings of length 12.

14. How many begin with 110?
15. How many begin with 11 and end with 10?
16. How many begin with 11 or end with 10?
17. How many have exactly four 1's?
18. How many have exactly four 1's and none of these 1's are adjacent to each other?
19. How many permutations of the seven letters *A, B, C, D, E, F, G* are there?
20. How many permutations of the seven letters *A, B, C, D, E, F, G* have *E* in the first position?
21. How many permutations of the seven letters *A, B, C, D, E, F, G* have *E* in one of the first two positions?
22. How many permutations of the seven letters *A, B, C, D, E, F, G* do not have vowels on the ends?
23. How many permutations of the seven letters *A, B, C, D, E, F, G* have the two vowels before the five consonants?
24. How many permutations of the seven letters *A, B, C, D, E, F, G* have *A* immediately to the left of *E*?
25. How many permutations of the seven letters *A, B, C, D, E, F, G* neither begin nor end with *A*?
26. How many permutations of the seven letters *A, B, C, D, E, F, G* do not have the vowels next to each other?
27. How many 8-element DNA sequences start with *C* and end with *C*?
28. How many 8-element DNA sequences contain exactly four *C*'s?
29. How many 5-element DNA sequences use all four bases, *A, T, C,* and *G*?
30. How many 8-element DNA sequences contain exactly two of the four bases?

In questions 31–37 nine people (*Ann, Ben, Cal, Dot, Ed, Fran, Gail, Hal,* and *Ida*) are in a room. Five of them stand in a row for a picture.

31. In how many ways can this be done if *Ben* is to be in the picture?
32. In how many ways can this be done if both *Ed* and *Gail* are in the picture?
33. In how many ways can this be done if neither *Ed* nor *Fran* are in the picture?
34. In how many ways can this be done if *Dot* is on the left end and *Ed* is on the right end?
35. In how many ways can this be done if *Hal* or *Ida* (but not both) are in the picture?
36. In how many ways can this be done if *Ed* and *Gail* are in the picture, standing next to each other?

37. In how many ways can this be done if Ann and Ben are in the picture, but not standing next to each other?
38. In a technician's box there are 400 VLSI chips, 12 of which are faulty. How many ways are there to pick two chips, so that one is a working chip and the other is faulty? (Assume that no chips are identical.)
39. How many truth tables are possible for compound propositions with the five variables p, q, r, s, t ?

In questions 40–44 let A be the set of all bit strings of length 10.

40. How many bit strings of length 10 are there?
41. How many bit strings of length 10 begin with 1101?
42. How many bit strings of length 10 have exactly six 0's?
43. How many bit strings of length 10 have equal numbers of 0's and 1's?
44. How many bit strings of length 10 have more 0's than 1's?

In questions 45–48 suppose you have 30 books (15 novels, 10 history books, and 5 math books). Assume that all 30 books are different. In how many ways can you

45. In how many ways can you put the 30 books in a row on a shelf?
46. In how many ways can you get a bunch of four books to give to a friend?
47. In how many ways can you get a bunch of three history books and seven novels to give to a friend?
48. In how many ways can you put the 30 books in a row on a shelf if the novels are on the left, the math books are in the middle, and the history books are on the right?
49. A class consists of 20 sophomores and 15 freshmen. The class needs to form a committee of size five.
- (a) How many committees are possible?
- (b) How many committees are possible if the committee must have three sophomores and two freshmen?

In questions 50–53 a club with 20 women and 17 men needs to form a committee of size six.

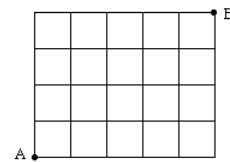
50. How many committees are possible?
51. How many committees are possible if the committee must have three women and three men?
52. How many committees are possible if the committee must have at least two men?
53. How many committees are possible if the committee must consist of all women or all men?
54. A club with 20 women and 17 men needs to choose three different members to be president, vice president, and treasurer.
- (a) In how many ways is this possible?
- (b) In how many ways is this possible if women will be chosen as president and vice president and a man as treasurer?
55. A class consists of 20 sophomores and 15 freshmen. The club needs to choose four different members to be president, vice president, secretary, and treasurer.
- (a) In how many ways is this possible?
- (b) In how many ways is this possible if sophomores will be chosen as president and treasurer and freshmen as vice president and secretary?
56. Suppose $|A| = 4$ and $|B| = 10$. Find the number of functions $f: A \rightarrow B$.
57. Suppose $|A| = 4$ and $|B| = 10$. Find the number of 1-1 functions $f: A \rightarrow B$.
58. Suppose $|A| = 10$ and $|B| = 4$. Find the number of 1-1 functions $f: A \rightarrow B$.

In questions 59–62 let A be the set of all strings of decimal digits of length five. For example 00312 and 19483 are strings in A .

59. Find $|A|$.
60. How many strings in A begin with 774?
61. How many strings in A have exactly one 5?
62. How many strings in A have exactly three 5s?
63. Make up a word problem in good English whose answer is $15!/10!$.
64. Make up a word problem in good English whose answer is $\binom{15}{4} \cdot \binom{7}{3}$.
65. How many subsets with an odd number of elements does a set with 10 elements have?
66. How many subsets with more than two elements does a set with 100 elements have?
67. Each user has a password 6 characters long where each character is an uppercase letter, a lowercase letter, or a digit. Each password must contain at least one digit. How long will it take to check every possible character combination, if each check takes one unit of time.
- In questions 68–71 suppose you have a class with 30 students — 10 freshmen, 12 sophomores, and 8 juniors.
68. In how many ways can you put all 30 in a line?
69. In how many ways can you put all students in a line so that the freshmen are first, the sophomores are in the middle, and the juniors are at the end?
70. In how many ways can you get a committee of 7?
71. In how many ways can you get a committee of 4 freshmen and 3 sophomores?
72. Using the ordinary alphabet and allowing repeated letters, find the number of words of length 8.
73. Using the ordinary alphabet and allowing repeated letters, find the number of words of length 8 that begin and end with T.
74. Using the ordinary alphabet and allowing repeated letters, find the number of words of length 8 that begin and end with the same letter.
75. Using the ordinary alphabet and allowing repeated letters, find the number of words of length 8 that have exactly one B.
76. Using the ordinary alphabet and allowing repeated letters, find the number of words of length 8 that have at least one C.
77. Using the ordinary alphabet and allowing repeated letters, find the number of words of length 8 that begin with L or end with R.
78. How many ways are there to select 6 students from a class of 25 to serve on a committee?
79. How many ways are there to select 6 students from a class of 25 to hold six different executive positions on a committee?
80. Find the number of subsets of $S = \{1, 2, 3, \dots, 10\}$ that contain the number 5.
81. Find the number of subsets of $S = \{1, 2, 3, \dots, 10\}$ that contain neither 5 nor 6.
82. Find the number of subsets of $S = \{1, 2, 3, \dots, 10\}$ that contain both 5 and 6.
83. Find the number of subsets of $S = \{1, 2, 3, \dots, 10\}$ that contain no odd numbers.
84. Find the number of subsets of $S = \{1, 2, 3, \dots, 10\}$ that contain exactly three elements.
85. Find the number of subsets of $S = \{1, 2, 3, \dots, 10\}$ that contain exactly three elements, one of which is 3.
86. Find the number of subsets of $S = \{1, 2, 3, \dots, 10\}$ that contain exactly five elements, all of them even.

- 87. Find the number of subsets of $S = \{1, 2, 3, \dots, 10\}$ that contain exactly three elements, all of them even.
- 88. Find the number of subsets of $S = \{1, 2, 3, \dots, 10\}$ with exactly five elements, two of which are 3 and 4.
- 89. Find the number of subsets of $S = \{1, 2, 3, \dots, 10\}$ with exactly five elements, including 3 or 4 but not both.
- 90. Find the number of subsets of $S = \{1, 2, 3, \dots, 10\}$ that contain exactly five elements, but neither 3 nor 4.
- 91. Find the number of subsets of $S = \{1, 2, 3, \dots, 10\}$ that contain exactly five elements, the sum of which is even.
- 92. Find the number of subsets of $S = \{1, 2, 3, \dots, 10\}$ that contain exactly four elements, the sum of which is odd.
- 93. Find the number of subsets of $S = \{1, 2, 3, \dots, 10\}$ that contain exactly four elements, the sum of which is even.
- 94. Suppose a restaurant serves a “special dinner” consisting of soup, salad, entree, dessert, and beverage. The restaurant has five kinds of soup, three kinds of salad, ten entrees, five desserts, and four beverages. How many different special dinners are possible? (Two special dinners are different if they differ in at least one selection.)

- 95. The figure at the right shows a 4-block by 5-block grid of streets. Find the number of ways in which you can go from point A to point B , where at each stage you can only go right or up. (You are not allowed to go left or down.) For example, one allowable route from A to B is:



Right, Right, Up, Right, Up, Up, Right, Right, Up.

- 96. Here is an incorrect solution to a problem. Find the error, explain why it is not correct, and give the correct answer.

“Problem: Find the number of ways to get two pairs of two different ranks (such as 2 jacks and 2 fives) in a 4-card hand from an ordinary deck of 52 cards.”

“Solution: There are 13 ways to get a rank (such as “kings”) for the first pair and $\binom{4}{2}$ ways to get a pair of that rank. Similarly, there are 12 ways to get a rank (such as “sevens”) for the second pair and $\binom{4}{2}$ ways to get a pair of that rank. Therefore there are $13 \cdot \binom{4}{2} \cdot 12 \cdot \binom{4}{2}$ ways to get 2 pairs.”
- 97. A game consisting of flipping a coin ends when the player gets two heads in a row, two tails in a row, or flips the coin four times.
 - (a) Draw a tree diagram to show the ways in which the game can end.
 - (b) In how many ways can the game end?
- 98. A factory makes automobile parts. Each part has a code consisting of a letter and three digits, such as C117, O076, or Z920. Last week the factory made 60,000 parts. Prove that there are at least three parts that have the same serial number.
- 99. A factory makes automobile parts. Each part has a code consisting of a digit, a letter, and a digit, with the digits distinct, such as 5C7, 1O6, or 3Z0. Last week the factory made 5,000 parts. Find the minimum number of parts that must have the same serial number.
- 100. Show that if five points are picked on or in the interior of a square of side length 2, then there are at least two of these points no farther than $\sqrt{2}$ apart.
- 101. A professor teaching a Discrete Math course gives a multiple choice quiz that has ten questions, each with four possible responses: a, b, c, d. What is the minimum number of students that must be in the professor’s class in order to guarantee that at least three answer sheets must be identical? (Assume that no answers are left blank.)
- 102. Show that in a group of ten people (where any two people are either friends or enemies) there are either three mutual friends or four mutual enemies.
- 103. A computer network consists of six computers. Each computer is directly connected to zero or more of the

other computers. Show that there are at least two computers in the network that are directly connected to the same number of computers.

104. A computer is programmed to print subsets of $\{1, 2, 3, 4, 5\}$ at random. If the computer prints 40 subsets, prove that some subset must have been printed at least twice.
105. A computer randomly prints three-digit codes, with no repeated digits in any code (for example, 387, 072, 760). What is the minimum number of codes that must be printed in order to guarantee that at least six of the codes are identical?
106. Explain how the Pigeonhole Principle can be used to show that among any 11 integers, at least two must have the same last digit.
107. Let s_1, s_2, \dots, s_{101} be 101 bit strings of length at most 9. Prove that there exist two strings, s_i and s_j , where $i \neq j$, that contain the same number of 0's and the same number of 1's. (For example, strings 001001 and 101000 contain the same number of 0's and the same number of 1's.)
108. You pick cards one at a time without replacement from an ordinary deck of 52 playing cards. What is the minimum number of cards you must pick in order to guarantee that you get
 (a) a pair (for example, two kings or two 5s).
 (b) three of a kind (for example, three 7s).
109. Use the binomial theorem to expand $(2a + b)^4$.
110. Use the binomial theorem to expand $(x + y)^5$.
111. Use the binomial theorem to expand $(a + 2)^6$.
112. Use the binomial theorem to expand $(2c - 3d)^4$.
113. Use the binomial theorem to expand $(x - \frac{3}{x})^5$.
114. Use the binomial theorem to expand $(x^2 + \frac{1}{x})^7$.
115. Use the binomial theorem to prove the following: $\binom{6}{0} + \binom{6}{1} + \dots + \binom{6}{6} = 2^6$.
116. Use the binomial theorem to prove the following:

$$\binom{100}{0} + \binom{100}{2} + \binom{100}{4} + \binom{100}{6} + \dots + \binom{100}{98} + \binom{100}{100} = \binom{100}{1} + \binom{100}{3} + \binom{100}{5} + \dots + \binom{100}{97} + \binom{100}{99}.$$
117. Use the binomial theorem to prove the following:

$$3^{100} = \binom{100}{0} + \binom{100}{1} \cdot 2 + \binom{100}{2} \cdot 2^2 + \binom{100}{3} \cdot 2^3 + \dots + \binom{100}{99} \cdot 2^{99} + \binom{100}{100} \cdot 2^{100}.$$
118. Find the coefficient of x^7y^5 in the expansion of $(3x - y)^{12}$.
119. Find the coefficient of x^5y^6 in the expansion of $(2x - y)^{11}$.
120. Find the coefficient of x^8 in the expansion of $(x^2 + 2)^{13}$.
121. Find the coefficient of x^9 in the expansion of $(2 + x^3)^{10}$.
122. Find the coefficient of x^5 in $(2 + x^2)^{12}$.
123. Find the number of terms in the expansion of $(5a + 8b)^{15}$.
124. Find the largest coefficient in the expansion of $(x + 1)^6$.
125. Find the largest coefficient in the expansion of $(x + 3)^5$.
126. List the derangements of 1, 2, 3, 4.
127. Find the number of positive integers not exceeding 1000 that are not divisible by 4, 6, or 9.
128. How many permutations of 12345 are there that leave 3 in the third position but leave no other integer in its

own position?

129. (a) Find the number of solutions to $x + y + z = 32$, where x , y , and z are nonnegative integers.
(b) Answer part (a), but assume that $x \geq 7$ and $y \geq 15$.
130. You have 20 pennies, 30 nickels, and 40 dimes. Assume that the pennies are identical, the nickels are identical, and the dimes are identical. In how many ways can you put all the coins in a row?
131. Find the number of permutations of the letters in the word CORRECT.
132. Find the number of permutations of the letters in the word COEFFICIENT.
133. Find the number of permutations of the letters in the word TATTERED.
134. Find the number of permutations of the letters in your last name.
135. How many different strings can be made using all the letters in the word GOOGOL?
136. (a) In how many ways are there to arrange the letters of the word NONSENSE?
(b) How many of these ways start or end with the letter O?
137. A doughnut shop sells 30 kinds of doughnuts. In how many ways can you
(a) get a bag of 12 doughnuts?
(b) get a bag of 12 doughnuts if you want at least 3 glazed doughnuts and at least 4 raspberry doughnuts?
(c) get a bag of 12 doughnuts if you want exactly 3 glazed doughnuts and exactly 4 raspberry doughnuts?
138. You have 50 of each of the following kinds of jellybeans: red, orange, green, yellow. The jellybeans of each color are identical.
(a) In how many ways can you put all the jellybeans in a row?
(b) How many handfuls of 12 are possible?

In 139–142 assume that you have 50 pennies and three jars, labeled A , B , and C .

139. In how many ways can you put the pennies in the jars, assuming that the pennies are distinguishable?
140. In how many ways can you put the pennies in the jars, assuming that the pennies are identical?
141. In how many ways can you put the pennies in the jars, assuming that the pennies are identical and each jar must have at least two pennies put into it?
142. In how many ways can you put the pennies in the jars, assuming that the pennies are identical and each jar must have an even number of pennies put into it?

In questions 143–146 assume that you have a bowl containing hard candies: 50 cherry, 50 strawberry, 40 orange, 70 lemon, and 40 pineapple. Assuming that the pieces of each flavor are identical,

143. How many handfuls of 15 are possible?
144. How many handfuls of 15 are possible with at least one piece of each flavor?
145. How many handfuls of 15 are possible with at least two pieces of each flavor?
146. How many handfuls of 15 are possible with at least three pieces of each flavor?
147. You have a pile of 20 identical blank cards. On each card you draw a circle, a plus, or a square. How many piles of 20 cards are possible?
148. You have 20 cards and 12 envelopes (labeled $1, 2, \dots, 12$). In how many ways can you put the 20 cards into the envelopes if
(a) the cards are distinct.
(b) the cards are identical.
(c) the cards are identical and no envelope can be left empty.

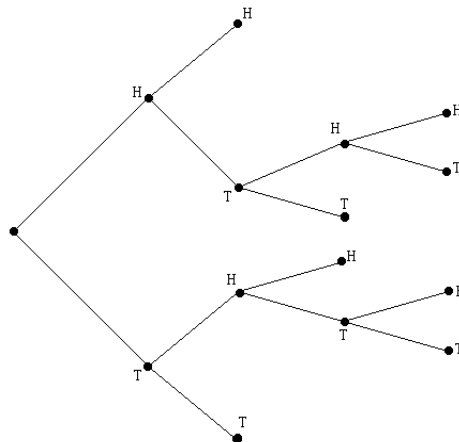
149. If the permutations of 1, 2, 3, 4, 5, 6 are written in lexicographic order, with 123456 in position #1, 123465 in position #2, etc., find the permutation immediately after 246531.
150. If the permutations of 1, 2, 3, 4, 5, 6 are written in lexicographic order, with 123456 in position #1, 123465 in position #2, etc., find the permutation immediately before 534126.
151. If the permutations of 1, 2, 3, 4, 5, 6 are written in lexicographic order, with 123456 in position #1, 123465 in position #2, etc., find the permutation in position #483.
152. Find the next largest permutation in lexicographic order after 1324.
153. Find the next largest permutation in lexicographic order after 52143.
154. Find the next largest permutation in lexicographic order after 6714235.
155. Find the next largest permutation in lexicographic order after 3254781.
156. Find the next four largest 4-combinations of the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$ after $\{1, 2, 3, 5\}$.

Answers for Chapter 6

1. 26^7 .
2. 26^6 .
3. 26^5 .
4. $2 \cdot 26^6$.
5. $26^6 + 26^6 - 26^5$.
6. $4 \cdot 26^5$.
7. $2 \cdot 26^6 + 2 \cdot 26^6 - 4 \cdot 26^5$.
8. $25 \cdot 26^5$.
9. $5 \cdot 26^6 + 5 \cdot 26^6 - 25 \cdot 26^5$.
10. $3 \cdot 26^4$.
11. 21^7 .
12. $5 \cdot 7 \cdot 21^6$.
13. First count the number of words that contain both A and B . This number is $8 \cdot 7 \cdot P(24, 6)$. Therefore the answer is equal to total number of words of length eight minus the number of words of length eight that have both A and B : $P(26, 8) - 8 \cdot 7 \cdot P(24, 6)$.
14. 2^9 .
15. 2^8 .
16. $2 \cdot 2^{10} - 2^8$.
17. $\binom{12}{4}$.
18. $\binom{9}{4}$.
19. $7!$.
20. $6!$.
21. $2 \cdot 6!$.
22. $5 \cdot 4 \cdot 5!$.
23. $2 \cdot 5!$.
24. $6!$.
25. $5 \cdot 6!$.

26. $7! - 2 \cdot 6!$.
27. 4^6
28. 5670
29. $4 \cdot 5 \cdot 4 \cdot 3 = 240$
30. 1524
31. $5P(8, 4)$.
32. $5 \cdot 4 \cdot P(7, 3)$.
33. $P(7, 5)$.
34. $P(7, 3)$.
35. $2 \cdot 5 \cdot P(7, 4)$.
36. $2 \cdot 4 \cdot P(7, 3)$.
37. $5 \cdot 4 \cdot P(7, 3) - 2 \cdot 4 \cdot P(7, 3)$.
38. $388 \cdot 12$.
39. 2^{2^5} .
40. 2^{10} .
41. 2^6 .
42. $\binom{10}{6}$.
43. $\binom{10}{5}$.
44. $\binom{10}{6} + \binom{10}{7} + \cdots + \binom{10}{10} = \frac{2^{10} - \binom{10}{5}}{2}$.
45. $30!$.
46. $\binom{30}{4}$.
47. $\binom{10}{3} \binom{15}{7}$.
48. $15! \cdot 5! \cdot 10!$.
49. (a) $\binom{35}{5}$. (b) $\binom{20}{3} \binom{15}{2}$.
50. $\binom{37}{6}$.
51. $\binom{20}{3} \binom{17}{3}$.
52. $\binom{17}{2} \binom{20}{4} + \binom{17}{3} \binom{20}{3} + \binom{17}{4} \binom{20}{2} + \binom{17}{5} \binom{20}{1} + \binom{17}{6} \binom{20}{0}$.
53. $\binom{20}{6} + \binom{17}{6}$.
54. (a) $37 \cdot 36 \cdot 35$. (b) $20 \cdot 19 \cdot 17$.
55. (a) $35 \cdot 34 \cdot 33 \cdot 32$. (b) $20 \cdot 19 \cdot 15 \cdot 14$.
56. 10^4 .
57. $P(10, 4)$.
58. 0.
59. 10^5 .
60. 10^2 .
61. $5 \cdot 9^4$.
62. $\binom{5}{3} \cdot 9^2$.
63. In how many ways can 5 out of 15 people be put in a row for a picture?
64. A class has 15 women and 7 men. In how many ways can a committee of 4 women and 3 men be formed?
65. $C(10, 1) + C(10, 3) + C(10, 5) + C(10, 7) + C(10, 9)$.
66. $2^{100} - C(100, 0) - C(100, 1) - C(100, 2)$.
67. $(26 + 26 + 10)^6 - (26 + 26)^6$ units of time.

68. $30!$.
69. $10! \cdot 12! \cdot 8!$.
70. $\binom{30}{7}$.
71. $\binom{10}{4} \binom{12}{3}$.
72. 26^8 .
73. 26^6 .
74. $26 \cdot 26^6$.
75. $8 \cdot 25^7$.
76. $26^8 - 25^8$.
77. $26^7 + 26^7 - 26^6$.
78. $C(25, 6)$.
79. $P(25, 6)$.
80. 2^9 .
81. 2^8 .
82. 2^8 .
83. 2^5 .
84. $C(10, 3)$.
85. $C(9, 2)$.
86. 1.
87. $C(5, 3)$.
88. $C(8, 3)$.
89. $2C(8, 4)$.
90. $C(8, 5)$.
91. $C(5, 1)C(5, 4) + C(5, 3)C(5, 2) + 1$.
92. $2C(5, 3)C(5, 1)$.
93. $2C(5, 4) + C(5, 2)^2$.
94. $5 \cdot 3 \cdot 10 \cdot 5 \cdot 4$.
95. $C(9, 5)$.
96. The same hand is counted twice. (For example, getting the kings of hearts and diamonds first and the sevens of clubs and hearts second is the same as getting the pair of sevens first and the pair of kings second.) To obtain the correct answer, divide the given answer by two.
97. (a)



- (b) 8.
98. The number of codes is $26 \cdot 10^3 = 26,000$. Since $\lceil 60,000/26,000 \rceil = 3$, at least three parts have the same code number.
99. The number of codes is $10 \cdot 26 \cdot 9 = 2,340$. Since $\lceil 6,000/2,340 \rceil = 3$, at least three parts have the same code number.
100. Divide the square into four congruent 1×1 squares. At least two of the five points lie in or on the edge of one of these 1×1 squares. The maximum distance between these two points is $\sqrt{2}$.
101. There are 4^{10} possible answer sheets. Therefore $2 \cdot 4^{10} + 1$ is the minimum number that will guarantee three identical answer sheets.
102. Let A be one of the people. A either has at least four friends or else has at least six enemies among the other nine people. Case 1: A has at least four friends, say B, C, D, E . If any two of B, C, D, E are friends, then these two together with A form a group of three mutual friends. If none of B, C, D, E are friends with each other, then B, C, D, E are four mutual enemies. Case 2: A has at least six enemies, say B, C, D, E, F, G . Applying the Pigeonhole Principle to this set of six, there are either three mutual friends or three mutual enemies. If there are three friends, we are done. If there are three mutual enemies, then these three together with A form a group of four mutual enemies.
103. Each computer can be connected to 0, 1, 2, 3, 4, or 5 other computers, but it is not possible in the network to have a computer connected to 0 others and a computer connected to all 5 others. Therefore there are only five possible connection numbers, which is smaller than the number of computers. By the Pigeonhole Principle at least two must have the same number of connections.
104. There are $2^5 = 32$ subsets. If 33 or more subsets are printed, at least one will have been printed twice.
105. There are $10 \cdot 9 \cdot 8 = 720$ different codes. Therefore $5 \cdot 720 + 1 = 3,601$ is the minimum number of printed codes that guarantees that at least six identical codes will be printed.
106. Use the eleven integers as the pigeons and the ten possible last digits as the pigeonholes.
107. There are ten possible lengths a bit string can have — 0, 1, 2, \dots , 9. Since there are 101 bit strings, there is a length number k such that at least 11 bit strings have length k . The number of 0's in these 11 bit strings must be one of the ten numbers 0, 1, 2, \dots , 9. Therefore, there are at least two bit strings s_i and s_j , with the same number of 0's. Since s_i and s_j have the same length, k , they both have the same number of 1's.
108. (a) 14. (b) 27.
109. $16a^4 + 32a^3b + 24a^2b^2 + 8ab^3 + b^4$.
110. $x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$.
111. $a^6 + 12a^5 + 60a^4 + 160a^3 + 240a^2 + 192a + 64$.
112. $16c^4 - 96c^3d + 216c^2d^2 - 216cd^3 + 81d^4$.
113. $x^5 - 15x^3 + 90x - 270/x + 405/x^3 - 243/x^5$.
114. $x^{14} + 7x^{11} + 21x^8 + 35x^5 + 35x^2 + 21/x + 7/x^4 + 1/x^7$.
115. In $(a + b)^n$ use $n = 6$, $a = b = 1$.
116. In $(a + b)^n$ use $n = 100$, $a = 1$, $b = -1$.
117. In $(a + b)^n$ use $n = 100$, $a = 1$, $b = 2$.
118. $-\binom{12}{7}3^7$.
119. $\binom{11}{5}2^5$.
120. $\binom{13}{9}2^9$.
121. $\binom{10}{3}2^7$.
122. 0.
123. 16.
124. 20.
125. 405.

126. 2143, 2341, 2413, 3142, 3412, 3421, 4123, 4312, 4321.
127. 611.
128. 9.
129. (a) $\binom{34}{2}$. (b) $\binom{12}{2}$.
130. $\frac{90!}{20! \cdot 30! \cdot 40!}$.
131. $\frac{7!}{2! \cdot 2!}$.
132. $\frac{11!}{(2!)^4}$.
133. $\frac{8!}{3! \cdot 2!}$.
134. Depends on last name.
135. $\frac{6!}{3!2!1!}$.
136. (a) $\frac{8!}{3!2!2!1!}$. (b) $\frac{2 \cdot 7!}{3!2!2!}$.
137. (a) $\binom{41}{29}$. (b) $\binom{34}{29}$. (c) $\binom{32}{27}$.
138. (a) $\frac{200!}{50!^4}$. (b) $\binom{15}{3}$.
139. 3^{50} .
140. $\binom{52}{2}$.
141. $\binom{46}{2}$.
142. $\binom{27}{2}$.
143. $\binom{19}{4}$.
144. $\binom{14}{4}$.
145. $\binom{9}{4}$.
146. $\binom{4}{4}$.
147. $\binom{22}{2}$.
148. (a) 12^{20} . (b) $\binom{31}{11}$. (c) $\binom{19}{11}$.
149. 251346.
150. 532641.
151. 512436.
152. 1342.
153. 52314.
154. 6714253.
155. 3254817.
156. $\{1, 2, 3, 6\}$, $\{1, 2, 3, 7\}$, $\{1, 2, 3, 8\}$, $\{1, 2, 4, 5\}$.

Questions for Chapter 7

1. What is the probability that a card chosen from an ordinary deck of 52 cards is an ace?
2. What is the probability that a randomly selected integer chosen from the first 100 positive integers is odd?
3. What is the probability that a randomly selected day of the year (366 days) is in May?

4. What is the probability that the sum of the numbers on two dice is even when they are rolled?
5. What is the probability that a fair coin lands Heads 6 times in a row?
6. What is the probability that a fair coin lands Heads 4 times out of 5 flips?
7. Three coins are tossed.
 - (a) List the elements in the sample space.
 - (b) Find the probability that exactly two heads show.
8. Suppose you pick two cards, one at a time, at random, from an ordinary deck of 52 cards. Find
 - (a) $p(\text{both cards are diamonds})$.
 - (b) $p(\text{the cards form a pair})$.
9. Suppose you and a friend each choose at random an integer between 1 and 8, inclusive. For example, some possibilities are (3, 7), (7, 3), (4, 4), (8, 1), where your number is written first and your friend's number second. Find the following probabilities:
 - (a) $p(\text{you pick 5 and your friend picks 8})$.
 - (b) $p(\text{sum of the two numbers picked is } < 4)$.
 - (c) $p(\text{both numbers match})$.
 - (d) $p(\text{the sum of the two numbers is a prime})$.
 - (e) $p(\text{your number is greater than your friend's number})$.
10. Prove or disprove: $p(E \cup F) = p(E) + p(F)$ for all events E and F .
11. Find and correct the error in the solution to the following problem:

Problem: You flip two coins and want to find the probability that both coins show heads.
Solution: There are three possible outcomes: 2 heads, 2 tails, or 1 head and 1 tail. Since a "success" is one of these three outcomes, $p(\text{both heads}) = 1/3$.
12. Let A be the set of all strings of decimal digits of length 5. For example 00312 and 19483 are two strings in A . You pick a string from A at random. What is the probability that
 - (a) the string begins with 7575.
 - (b) the string has no 4 in it.

In questions 13–16 you have 40 different books (20 math books, 15 history books, and 5 geography books).

13. You pick one book at random. What is the probability that the book is a history book?
 14. You pick one book at random. What is the probability that the book is not a geography book?
 15. You pick two books at random, one at a time. What is the probability that both books are history books?
 16. You pick two books at random, one at a time. What is the probability that the two books are from different disciplines?
- In questions 17–19 suppose you have a class with 30 students — 10 freshmen, 12 sophomores, and 8 juniors.
17. You pick one student at random. What is the probability that the student is not a junior?
 18. You pick two students at random, one at a time. What is the probability that both are freshmen?
 19. You pick two students at random, one at a time. What is the probability that the second student is a freshman, given that the first is a freshman?
 20. In a certain lottery game, three distinct numbers between 10 and 25 (inclusive) are chosen as the winning numbers. What is the probability that the winning numbers are all composite numbers.
 21. In a certain lottery game you choose a set of six numbers out of 54 numbers. Find the probability that none of your numbers match the six winning numbers.

In questions 22–27 you pick a bit string from the set of all bit strings of length ten.

22. What is the probability that the bit string has exactly two 1's?
23. What is the probability that the bit string has exactly two 1's, given that the string begins with a 1?

24. What is the probability that the bit string begins and ends with 0?
25. What is the probability that the bit string has more 0's than 1's?
26. What is the probability that the bit string has the sum of its digits equal to seven?
27. What is the probability that the bit string begins with 111?
28. A group of ten women and ten men are in a room. If five of the 20 are selected at random and put in a row for a picture, what is the probability that the five are of the same sex?
29. A group of ten women and ten men are in a room. A committee of four is chosen at random. Find the probability that the committee consists only of women?
30. You pick a word at random from the set of all words of length six of letters of the alphabet with no repeated letters. What is the probability that the word has exactly one vowel?
31. You pick a word at random from the set of all words of length six of letters of the alphabet with no repeated letters. What is the probability that the word begins and ends with a vowel?
32. A red and a green die are rolled. What is the probability of getting a sum of six, given that the number on the red die is even.
33. A red and a green die are rolled. What is the probability of getting a sum of six, given that the number on the green die is odd?

In 34–39 an experiment consists of picking at random a bit string of length five. Consider the following events:

E_1 : the bit string chosen begins with 1;

E_2 : the bit string chosen ends with 1;

E_3 : the bit string chosen has exactly three 1's.

34. Find $p(E_1|E_3)$.
35. Find $p(E_3|E_2)$.
36. Find $p(E_2|E_3)$.
37. Find $p(E_3|E_1 \cap E_2)$.
38. Determine whether E_1 and E_2 are independent.
39. Determine whether E_2 and E_3 are independent.

In questions 40–42 you flip a biased coin, where $p(\text{heads}) = 3/4$ and $p(\text{tails}) = 1/4$, ten times.

40. Find $p(\text{exactly 9 heads})$.
41. Find $p(\text{exactly 7 heads})$.
42. Find $p(\text{at least 7 heads})$.
43. Urn 1 contains 2 blue tokens and 8 red tokens; urn 2 contains 12 blue tokens and 3 red tokens. You pick an urn at random and draw out a token at random from that urn. Given that the token is blue, what is the probability that the token came from urn 1?
44. Urn 1 contains 2 blue tokens and 8 red tokens; urn 2 contains 12 blue tokens and 3 red tokens. You roll a die to determine which urn to choose: if you roll a 1 or 2 you choose urn 1; if you roll a 3, 4, 5, or 6 you choose urn 2. Once the urn is chosen, you draw out a token at random from that urn. Given that the token is blue, what is the probability that the token came from urn 1?

In questions 45–47 a bowl has eight ping pong balls numbered 1,2,2,3,4,5,5,5. You pick a ball at random.

45. Find $p(\text{the number on the ball drawn is } \geq 3)$.
46. Find $p(\text{the number on the ball drawn is even})$.

47. Find $E(X)$, where X = the number on the ball you draw.
48. A die has the numbers 1, 2, 2, 3, 3, 3 on its six sides. If the die is rolled, what is the expected value and variance of the number showing?
49. A pair of dice, each with the numbers 1, 2, 2, 3, 3, 3 on its six sides are rolled.
(a) What is the expected value of the sum of the numbers showing?
(b) What is the expected value of the product of the numbers showing in part (a)?
50. You have seven cards, numbered 3 through 9, and you pick one at random. If you pick a card with a prime number, you get 1 point; if you pick a card with a composite number, you lose 1 point. Find the expected value of the number of points you get.
51. You flip a coin. If it lands heads, you lose 1 point. If it lands tails, you flip the coin again, and lose 1 point if it lands heads and get 3 points if it lands tails. What is the expected value of the number of points you get when you play this game.
52. Each of 26 cards has a different letter of the alphabet on it. You pick one card at random. A vowel is worth 3 points and a consonant is worth 0 points. Let X = the value of the card picked. Find $E(X)$, $V(X)$, and the standard deviation of X .
53. You have two decks of 26 cards. Each card in each of the two decks has a different letter of the alphabet on it. You pick at random one card from each of the two decks. A vowel is worth 3 points and a consonant is worth 0 points. Let X = the sum of the values of the two cards picked. Find $E(X)$, $V(X)$, and the standard deviation of X .

Answers for Chapter 7

1. $4/52$.
2. $50/100$.
3. $31/366$.
4. $18/36$.
5. $1/2^6$.
6. $C(5, 4)/2^5 = 5/32$.
7. (a) HHH, HHT, HTH, THH, HTT, THT, TTH, TTT. (b) $3/8$.
8. (a) $(13/52)(12/51)$. (b) $3/51$.
9. (a) $1/64$. (b) $3/64$. (c) $8/64$. (d) $23/64$. (e) $28/64$.
10. False. Choose one card at random from a deck of 52 cards. Let E = choose a diamond, F = choose a king. Then $p(E \cup F) = 16/52$ while $p(E) + p(F) = 17/52$.
11. The probabilities of the three outcomes are not equal. Using $\{HH, HT, TH, TT\}$ as the sample space, the correct answer, $1/4$, is obtained.
12. (a) $10/10^5$. (b) $9^5/10^5$.
13. $15/40$.
14. $35/40$.
15. $(15 \cdot 14)/(40 \cdot 39)$.
16. $1 - \frac{20 \cdot 19 + 15 \cdot 14 + 5 \cdot 4}{40 \cdot 39}$.
17. $22/30$.
18. $(10 \cdot 9)/(30 \cdot 29)$.
19. $9/29$.

20. $\binom{11}{3}/\binom{16}{3}$.
21. $\binom{48}{6}/\binom{54}{6}$.
22. $\binom{10}{2}/2^{10}$.
23. $9/2^{10}$.
24. $2^8/2^{10}$.
25. $(\binom{10}{6} + \binom{10}{7} + \binom{10}{8} + \binom{10}{9} + \binom{10}{10})/2^{10}$.
26. $\binom{10}{7}/2^{10}$.
27. $2^7/2^{10}$.
28. $\frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16} + \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16}$.
29. $\binom{10}{4}/\binom{20}{4}$.
30. $5 \cdot 6 \cdot P(21, 5)/P(26, 6)$.
31. $5 \cdot 4 \cdot P(24, 4)/P(26, 6)$.
32. $1/9$.
33. $1/6$.
34. $6/10$.
35. $6/16$.
36. $6/10$.
37. $3/8$.
38. Yes.
39. No.
40. $\binom{10}{9} \left(\frac{3}{4}\right)^9 \frac{1}{4}$.
41. $\binom{10}{7} \left(\frac{3}{4}\right)^7 \left(\frac{1}{4}\right)^3$.
42. $\binom{10}{7} \left(\frac{3}{4}\right)^7 \left(\frac{1}{4}\right)^3 + \binom{10}{8} \left(\frac{3}{4}\right)^8 \left(\frac{1}{4}\right)^2 + \binom{10}{9} \left(\frac{3}{4}\right)^9 \frac{1}{4} + \binom{10}{10} \left(\frac{3}{4}\right)^{10}$.
43. $(\frac{2}{10} \cdot \frac{1}{2}) / (\frac{2}{10} \cdot \frac{1}{2} + \frac{12}{15} \cdot \frac{1}{2}) = 1/5$.
44. $(\frac{2}{10} \cdot \frac{1}{3}) / (\frac{2}{10} \cdot \frac{1}{3} + \frac{12}{15} \cdot \frac{2}{3}) = 1/9$.
45. $5/8$.
46. $3/8$.
47. $27/8$.
48. $7/3, 5/9$.
49. (a) $14/3$. (b) $49/9$.
50. $-1/7$.
51. 0 .
52. $15/26, 1.38, 1.18$.
53. $2 \cdot \frac{15}{26}, 2 \cdot 1.38, 1.66$.

Questions for Chapter 8

In questions 1–4, describe each sequence recursively. Include initial conditions and assume that the sequences begin with a_1 .

1. $a_n =$ the number of bit strings of length n with an even number of 0's.
2. $a_n =$ the number of bit strings of length n that begin with 1.
3. $a_n =$ the number of bit strings of length n that contain a pair of consecutive 0's.
4. $a_n =$ the number of ways to go down an n -step staircase if you go down 1, 2, or 3 steps at a time.

In questions 5–10 determine whether the recurrence relation is a linear homogeneous recurrence relation with constant coefficients.

5. $a_n = 0.7a_{n-1} - 0.3a_{n-2}$.
6. $a_n = na_{n-1}$.
7. $a_n = 5a_{n-1}^2 - 3a_{n-2}^2$.
8. $a_n = a_{n-3}$.
9. $a_n - 7a_{n-2} + a_{n-5} = 0$.
10. $a_n + a_{n-1} = 1$.
11. A vending machine dispensing books of stamps accepts only \$1 coins, \$1 bills, and \$2 bills. Let a_n denote the number of ways of depositing n dollars in the vending machine, where the order in which the coins and bills are deposited matters.
 - (a) Find a recurrence relation for a_n and give the necessary initial condition(s).
 - (b) Find an explicit formula for a_n by solving the recurrence relation in part (a).
12. Find the solution of the recurrence relation $a_n = 3a_{n-1}$ with $a_0 = 2$.

In questions 13–20 solve the recurrence relation either by using the characteristic equation or by discovering a pattern formed by the terms.

13. $a_n = 5a_{n-1} - 4a_{n-2}$, $a_0 = 1$, $a_1 = 0$.
14. $a_n = 5a_{n-1} - 4a_{n-2}$, $a_0 = 0$, $a_1 = 1$.
15. $a_n = -10a_{n-1} - 21a_{n-2}$, $a_0 = 2$, $a_1 = 1$.
16. $a_n = a_{n-2}$, $a_0 = 2$, $a_1 = -1$.
17. $a_n = 2a_{n-1} + 2a_{n-2}$, $a_0 = 0$, $a_1 = 1$.
18. $a_n = 3na_{n-1}$, $a_0 = 2$.
19. $a_n = a_{n-1} + 3n$, $a_0 = 5$.
20. $a_n = 2a_{n-1} + 5$, $a_0 = 3$.
21. $a_n = a_{n-1} + 2n + 1$, $a_0 = 5$.
22. The solutions to $a_n = -3a_{n-1} + 18a_{n-2}$ have the form $a_n = c \cdot 3^n + d \cdot (-6)^n$. Which of the following are solutions to the given recurrence relation?

(a) $a_n = 3^{n+1} + (-6)^n$.	(b) $a_n = 5(-6)^n$.
(c) $a_n = 3c - 6d$.	(d) $a_n = 3^{n-2}$.
(e) $a_n = \pi(3^n + (-6)^n)$.	(f) $a_n = -3^n$.
(g) $a_n = 3^n(1 + (-2)^n)$.	(h) $a_n = 3^n + 6^n$.
23. Assume that the characteristic equation for a homogeneous linear recurrence relation with constant coefficients is $(r - 5)^3 = 0$. Describe the form for the general solution to the recurrence relation.
24. Assume that the characteristic equation for a homogeneous linear recurrence relation with constant coefficients

is $(r + 2)(r + 4)^2 = 0$. Describe the form for the general solution to the recurrence relation.

25. Assume that the characteristic equation for a homogeneous linear recurrence relation with constant coefficients is $(r + 1)^4(r - 1)^4 = 0$. Describe the form for the general solution to the recurrence relation.
26. Assume that the characteristic equation for a homogeneous linear recurrence relation with constant coefficients is $(r - 3)^2(r - 4)^3(r + 7)^2 = 0$. Describe the form for the general solution to the recurrence relation.
27. The Catalan numbers C_n count the number of strings of n +’s and n -’s with the following property: as each string is read from left to right, the number of +’s encountered is always at least as large as the number of -’s.
- Verify this by listing these strings of lengths 2, 4, and 6 and showing that there are C_1 , C_2 , and C_3 of these, respectively.
 - Explain how counting these strings is the same as counting the number of ways to correctly parenthesize strings of variables.
28. What form does a particular solution of the linear nonhomogeneous recurrence relation $a_n = 4a_{n-1} - 4a_{n-2} + F(n)$ have when $F(n) = 2^n$?
29. What form does a particular solution of the linear nonhomogeneous recurrence relation $a_n = 4a_{n-1} - 4a_{n-2} + F(n)$ have when $F(n) = n2^n$?
30. What form does a particular solution of the linear nonhomogeneous recurrence relation $a_n = 4a_{n-1} - 4a_{n-2} + F(n)$ have when $F(n) = n^2 \cdot 4^n$?
31. What form does a particular solution of the linear nonhomogeneous recurrence relation $a_n = 4a_{n-1} - 4a_{n-2} + F(n)$ have when $F(n) = (n^2 + 1)2^n$?
32. Consider the recurrence relation $a_n = 2a_{n-1} + 3n$.
- Write the associated homogeneous recurrence relation.
 - Find the general solution to the associated homogeneous recurrence relation.
 - Find a particular solution to the given recurrence relation.
 - Write the general solution to the given recurrence relation.
 - Find the particular solution to the given recurrence relation when $a_0 = 1$.
33. Consider the recurrence relation $a_n = -a_{n-1} + n$.
- Write the associated homogeneous recurrence relation.
 - Find the general solution to the associated homogeneous recurrence relation.
 - Find a particular solution to the given recurrence relation.
 - Write the general solution to the given recurrence relation.
 - Find the particular solution to the given recurrence relation when $a_0 = 1$.
34. Consider the recurrence relation $a_n = 3a_{n-1} + 5^n$.
- Write the associated homogeneous recurrence relation.
 - Find the general solution to the associated homogeneous recurrence relation.
 - Find a particular solution to the given recurrence relation.
 - Write the general solution to the given recurrence relation.
 - Find the particular solution to the given recurrence relation when $a_0 = 1$.
35. Consider the recurrence relation $a_n = 2a_{n-1} + 1$.
- Write the associated homogeneous recurrence relation.
 - Find the general solution to the associated homogeneous recurrence relation.
 - Find a particular solution to the given recurrence relation.
 - Write the general solution to the given recurrence relation.
 - Find the particular solution to the given recurrence relation when $a_0 = 1$.
36. Suppose $f(n) = 3f(n/2) + 1$, $f(1) = 1$. Find $f(8)$.
37. Suppose $f(n) = f(n/3) + 2n$, $f(1) = 1$. Find $f(27)$.
38. Suppose $f(n) = 2f(n/2)$, $f(8) = 2$. Find $f(1)$.

39. Suppose $f(n) = 2f(n/2) + 3$, $f(16) = 51$. Find $f(2)$.
40. Suppose $f(n) = 4f(n/2) + n + 2$, $f(1) = 2$. Find $f(8)$.
41. Use generating functions to solve $a_n = 3a_{n-1} + 2^n$, $a_0 = 5$.
42. Use generating functions to solve $a_n = 5a_{n-1} + 1$, $a_0 = 1$.

In questions 43–52 write the first seven terms of the sequence determined by the generating function.

43. $(x + 3)^2$.
44. $(1 + x)^5$.
45. $(1 + x)^9$.
46. $1/(1 - 3x)$.
47. $x^2/(1 - x)$.
48. $(1 + x)/(1 - x)$.
49. 5.
50. $e^x + e^{-x}$.
51. $\cos x$.
52. $\frac{1}{1 - x} - x^2 - x^3$.

In questions 53–63 find the coefficient of x^8 in the power series of each of the function.

53. $(1 + x^2 + x^4)^3$.
54. $(1 + x^2 + x^4 + x^6)^3$.
55. $(1 + x^2 + x^4 + x^6 + x^8)^3$.
56. $(1 + x^2 + x^4 + x^6 + x^8 + x^{10})^3$.
57. $(1 + x^3)^{12}$.
58. $(1 + x)(1 + x^2)(1 + x^3)(1 + x^4)(1 + x^5)$.
59. $1/(1 - 2x)$.
60. $x^3/(1 - 3x)$.
61. $1/(1 - x)^2$.
62. $x^2/(1 + 2x)^2$.
63. $1/(1 - 3x^2)$.

In questions 64–76 find a closed form for the generating function for the sequence.

64. 4, 8, 16, 32, 64, ...
65. 1, 0, 1, 0, 1, 0, 1, 0, ...
66. 2, 0, 0, 2, 0, 0, 2, 0, 0, 2, ...
67. 2, 4, 6, 8, 10, 12, ...
68. 0, 0, 0, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, ...
69. 2, 3, 4, 5, 6, 7, ...
70. 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, ...

71. $1, -1, \frac{1}{2!}, -\frac{1}{3!}, \frac{1}{4!}, -\frac{1}{5!}, \dots$
72. $1, \frac{1}{2!}, \frac{1}{4!}, \frac{1}{6!}, \frac{1}{8!}, \dots$
73. $1, -1, 1, -1, 1, -1, 1, -1, \dots$
74. $1, 0, -1, 0, 1, 0, -1, 0, 1, 0, -1, \dots$
75. $\binom{50}{50}, \binom{50}{49}, \binom{50}{48}, \dots, \binom{50}{1}, \binom{50}{0}, 0, 0, 0, \dots$
76. $\binom{50}{1}, 2\binom{50}{2}, 3\binom{50}{3}, \dots, 50\binom{50}{50}, 0, 0, 0, \dots$
77. Set up a generating function and use it to find the number of ways in which eleven identical coins can be put in three distinct envelopes if each envelope has at least two coins in it.
78. Set up a generating function and use it to find the number of ways in which eleven identical coins can be put in three distinct envelopes if each envelope has most six coins in it.
79. Set up a generating function and use it to find the number of ways in which eleven identical coins can be put in three distinct envelopes if no envelope is empty.
80. Set up a generating function and use it to find the number of ways in which eleven identical coins can be put in three distinct envelopes if each envelope has an even number of coins in it.
81. Set up a generating function and use it to find the number of ways in which eleven identical coins can be put in three distinct envelopes if each envelope has at least two but no more than five coins in it.
82. Set up a generating function and use it to find the number of ways in which eleven identical coins can be put in three distinct envelopes (labeled A, B, C) if envelope A has at least three coins in it.
83. Set up a generating function and use it to find the number of ways in which eleven identical coins can be put in three distinct envelopes (labeled A, B, C) envelopes A and B have the same number of coins in them.
84. Set up a generating function and use it to find the number of ways in which nine identical blocks can be given to four children if each child gets at least one block.
85. Set up a generating function and use it to find the number of ways in which nine identical blocks can be given to four children, if each child gets at least two blocks.
86. Set up a generating function and use it to find the number of ways in which nine identical blocks can be given to four children, if each child gets at most five blocks.
87. Set up a generating function and use it to find the number of ways in which nine identical blocks can be given to four children, if the oldest child gets three blocks.
88. Set up a generating function and use it to find the number of ways in which nine identical blocks can be given to four children, if the oldest child gets at most three blocks.
89. Set up a generating function and use it to find the number of ways in which nine identical blocks can be given to four children, if the oldest child gets either 2 or 3 blocks.
90. If $G(x)$ is the generating function for $a_0, a_1, a_2, a_3, \dots$, describe in terms of $G(x)$ the generating function for $0, 0, 0, a_0, a_1, a_2, \dots$.
91. If $G(x)$ is the generating function for $a_0, a_1, a_2, a_3, \dots$, describe in terms of $G(x)$ the generating function for $0, 0, 0, a_3, a_4, a_5, \dots$.
92. If $G(x)$ is the generating function for $a_0, a_1, a_2, a_3, \dots$, describe in terms of $G(x)$ the generating function for $a_3, a_4, a_5, a_6, \dots$.
93. If $G(x)$ is the generating function for $a_0, a_1, a_2, a_3, \dots$, describe in terms of $G(x)$ the generating function for $a_0, 0, a_1, 0, a_2, 0, a_3, 0, a_4, \dots$.
94. If $G(x)$ is the generating function for $a_0, a_1, a_2, a_3, \dots$, describe in terms of $G(x)$ the generating function for

$a_0, 3a_1, 9a_2, 27a_3, 81a_4, \dots$

95. If $G(x)$ is the generating function for $a_0, a_1, a_2, a_3, \dots$, describe in terms of $G(x)$ the generating function for $a_0, 0, 0, a_1, 0, 0, a_2, 0, 0, a_3, \dots$.
96. If $G(x)$ is the generating function for $a_0, a_1, a_2, a_3, \dots$, describe in terms of $G(x)$ the generating function for $5, a_1, 0, a_3, a_4, a_5, \dots$.
97. Use generating functions to solve $a_n = 5a_{n-1} + 3$, $a_0 = 2$.
98. Use generating functions to solve $a_n = 7a_{n-1} - 10a_{n-2}$, $a_0 = 1$, $a_1 = 1$.
99. Use generating functions to solve $a_n = 3a_{n-1} + 2^n + 5$, $a_0 = 1$.
100. Find $|A_1 \cup A_2 \cup A_3 \cup A_4|$ if each set A_i has 100 elements, each intersection of two sets has 60 elements, each intersection of three sets has 20 elements, and there are 10 elements in all four sets.
101. Find $|A_1 \cup A_2 \cup A_3 \cup A_4|$ if each set A_i has 150 elements, each intersection of two sets has 80 elements, each intersection of three sets has 20 elements, and there are no elements in all four sets.
102. Suppose you use the principle of inclusion-exclusion to find the size of the union of four sets. How many terms must be added or subtracted?
103. Find the number of positive integers ≤ 1000 that are multiples of at least one of 3, 5, 11.
104. Find the number of positive integers ≤ 1000 that are multiples of at least one of 2, 6, 12.
105. Find the number of positive integers ≤ 1000 that are multiples of at least one of 3, 4, 12.
106. Suppose $|A| = |B| = |C| = 100$, $|A \cap B| = 60$, $|A \cap C| = 50$, $|B \cap C| = 40$, and $|A \cup B \cup C| = 175$. How many elements are in $A \cap B \cap C$?
107. How many positive integers not exceeding 1000 are not divisible by either 4 or 6?
108. A doughnut shop sells 20 kinds of doughnuts. You want to buy 30 doughnuts. How many possibilities are there if you want at most six of any one kind?
109. A doughnut shop sells 20 kinds of doughnuts. You want to buy 30 doughnuts. How many possibilities are there if you want at most 12 of any one kind?
110. A market sells ten kinds of soda. You want to buy 12 bottles. How many possibilities are there if you want
(a) at least one of each kind?
(b) at most seven bottles of any kind?
111. A market sells ten kinds of soda. You want to buy 12 bottles. How many possibilities are there? if you want at most three bottles of any kind?
112. Suppose you have 100 identical marbles and five jars (labeled A, B, C, D, E). In how many ways can you put the marbles in the jars if:
(a) each jar has at least six marbles in it?
(b) each jar has at most forty marbles in it?
113. How many ways are there to choose five donuts if there are eight varieties and only the type of each donut matters?
114. A market sells 40 kinds of candy bars. You want to buy 15 candy bars.
(a) How many possibilities are there?
(b) How many possibilities are there if you want at least three peanut butter bars and at least five almond bars?
(c) How many possibilities are there if you want exactly three peanut butter bars and exactly five almond bars?
(d) How many possibilities are there if you want at most four toffee bars and at most six mint bars?
115. How many permutations of all 26 letters of the alphabet are there that contain at least one of the words DOG,

BIG, OIL?

116. How many permutations of all 26 letters of the alphabet are there that contain at least one of the words CART, SHOW, LIKE?
117. How many permutations of all 26 letters of the alphabet are there that contain at least one of the words SWORD, PLANT, CARTS?
118. How many permutations of all 26 letters of the alphabet are there that contain none of the words: SAVE, PLAY, SNOW?
119. How many permutations of all 26 letters of the alphabet are there that contain at least one of the words: CAR, CARE, SCAR, SCARE?
120. How many permutations of the 26 letters of the alphabet are there that do not contain any of the following strings: LOP, SLOP, SLOPE, LOPE.
121. You have ten cards, numbered 1 through 10. In how many ways can you put the ten cards in a row so that card i is not in spot i , for $i = 1, 2, \dots, 10$?
122. Suppose $|A| = 8$ and $|B| = 4$. Find the number of functions $f : A \rightarrow B$ that are onto B .
123. An office manager has four employees and nine reports to be done. In how many ways can the reports be assigned to the employees so that each employee has at least one report to do.
124. An office manager has five employees and 12 projects to be completed. In how many ways can the projects be assigned to the employees so that each employee works on at least one project.
125. Find the number of ways to put eight different books in five boxes, if no box is allowed to be empty.
126. Find the number of bit strings of length eight that contain a pair of consecutive 0's.
127. Find the number of ways to climb a 12-step staircase, if you go up either one or three steps at a time.
128. Find the number of strings of 0's, 1's, and 2's of length six that have no consecutive 0's.

Answers for Chapter 8

1. $a_n = a_{n-1} + 2^{n-2}$, $a_1 = 1$.
2. $a_n = 2a_{n-1}$, $a_1 = 1$.
3. $a_n = a_{n-1} + a_{n-2} + 2^{n-2}$, $a_1 = 0$, $a_2 = 1$.
4. $a_n = a_{n-1} + a_{n-2} + a_{n-3}$, $a_1 = 0$, $a_2 = 1$, $a_3 = 1$.
5. Yes.
6. No.
7. No.
8. Yes.
9. Yes.
10. No.
11. (a) $a_n = 2a_{n-1} + a_{n-2}$, $a_0 = 1$, $a_1 = 2$. (b) $a_n = \alpha(1 + \sqrt{2})^n + \beta(1 - \sqrt{2})^n$ where $\alpha = (1 + \frac{1}{\sqrt{2}})/2$ and

$$\beta = (1 - \frac{1}{\sqrt{2}})/2.$$

12. $a_n = 2 \cdot 3^n$
13. $a_n = (-1/3) \cdot 4^n + (4/3) \cdot 1^n.$
14. $a_n = (1/3) \cdot 4^n - (1/3) \cdot 1^n.$
15. $a_n = (-7/4)(-7)^n + (15/4) \cdot (-3)^n.$
16. $a_n = (1/2) \cdot 1^n + (3/2) \cdot (-1)^n.$
17. $a_n = (\sqrt{3}/6)(1 + \sqrt{3})^n - (\sqrt{3}/6)(1 - \sqrt{3})^n.$
18. $a_n = 2 \cdot 3^n \cdot n!.$
19. $a_n = 5 + 3^{\frac{n(n+1)}{2}}.$
20. $a_n = 3 \cdot 2^n + 5(2^n - 1) = 2^{n+3} - 5.$
21. $a_n = 5 + n(n + 1) + n = n^2 + 2n + 5.$
22. (a) Yes. (b) Yes. (c) No. (d) Yes. (e) Yes. (f) Yes. (g) Yes. (h) No.
23. $a_n = c5^n + dn5^n + en^25^n.$
24. $a_n = c(-2)^n + d(-4)^n + en(-4)^n.$
25. $a_n = c(-1)^n + dn(-1)^n + en^2(-1)^n + fn^3(-1)^n + g + hn + in^2 + jn^3.$
26. $a_n = c3^n + dn3^n + en^23^n + f4^n + gn4^n + hn^24^n + i(-7)^n + jn(-7)^n.$
27. (a) $C_1: +- , C_2: +-+-, ++-- , C_3: +-+-- , +++--- , +-+--- , +-+--- , +-+--- .$
 (b) Treat each + as a left parenthesis and each - as a right parenthesis.
28. $p_0n^22^n.$
29. $n^2(p_1n + p_0)2^n.$
30. $(p_2n^2 + p_1n + p_0)4^n.$
31. $n^2(p_2n^2 + p_1n + p_0)2^n.$
32. (a) $a_n = 2a_{n-1}.$ (b) $a_n = c2^n.$ (c) $a_n = -3n-6.$ (d) $a_n = -3n-6+c2^n.$ (e) $a_n = -3n-6+7 \cdot 2^n.$
33. (a) $a_n = -a_{n-1}.$ (b) $a_n = c(-1)^n.$ (c) $a_n = \frac{n}{2} + \frac{1}{4}.$ (d) $a_n = \frac{n}{2} + \frac{1}{4} + c(-1)^n.$ (e) $a_n = \frac{n}{2} + \frac{1}{4} + \frac{3}{4}(-1)^n.$
34. (a) $a_n = 3a_{n-1}$ (b) $a_n = c3^n$ (c) $a_n = \frac{5^{n+1}}{2}$ (d) $a_n = \frac{5^{n+1}}{2} + c3^n$ (e) $a_n = \frac{5^{n+1}}{2} - \frac{3^{n+1}}{2}$
35. (a) $a_n = 2a_{n-1}.$ (b) $a_n = c2^n.$ (c) $a_n = -1.$ (d) $a_n = c2^n - 1.$ (e) $a_n = 2^{n+1} - 1.$
36. 40.
37. 79.
38. 1/4.
39. 15/4.
40. 226.
41. $a_n = 7 \cdot 3^n - 2 \cdot 2^n.$
42. $a_n = \frac{5^{n+1}}{4} - \frac{1}{4}.$
43. (a) 9, 6, 1, 0, 0, 0, 0.
44. 1, 5, 10, 10, 5, 1, 0.
45. 1, 9, 36, 84, 126, 126, 84.
46. 1, 3, 9, 27, 81, 243, 729.
47. 0, 0, 1, 1, 1, 1, 1.
48. 1, 2, 2, 2, 2, 2, 2.
49. 5, 0, 0, 0, 0, 0, 0.
50. 2, 0, 1, 0, $\frac{1}{12}, 0, \frac{1}{360}.$

51. $1, 0, \frac{-1}{2!}, 0, \frac{1}{4!}, 0, \frac{-1}{6!}$.
52. $1, 1, 0, 0, 1, 1, 1$.
53. 6.
54. 12.
55. 15.
56. 15.
57. 0.
58. 3.
59. 2^8 .
60. 3^5 .
61. 9.
62. $7 \cdot 2^6$.
63. 3^4 .
64. $\frac{4}{1-2x}$.
65. $\frac{1}{1-x^2}$.
66. $\frac{2}{1-x^3}$.
67. $\frac{2}{(1-x)^2}$.
68. $x^3(1+x+x^2+x^3)$.
69. $\frac{1}{(1-x)^2} + \frac{1}{1-x} = \frac{2-x}{(1-x)^2}$.
70. $\frac{1}{1-x} - \frac{1}{1-x^3}$.
71. e^{-x} .
72. e^{x^2} .
73. $\frac{1}{1+x}$.
74. $\frac{1}{1+x^2}$.
75. $(1+x)^{50}$.
76. $50(1+x)^{49}$.
77. $(x^2+x^3+x^4+\dots)^3$, 21.
78. $(1+x+x^2+\dots+x^6)^3$, 33.
79. $(x+x^2+x^3+\dots)^3$, 45.
80. $(1+x^2+x^4+x^6+\dots)^3$, 0.
81. $(x^2+x^3+x^4+x^5)^3$, 12.
82. $(x^3+x^4+x^5+x^6+\dots)(1+x+x^2+x^3+\dots)^2$, 45.
83. $(1+x^2+x^4+x^6+x^8+x^{10})(1+x+x^2+x^3+\dots)$, 6.
84. $(x+x^2+x^3+\dots)^4$, 56.
85. $(x^2+x^3+x^4+\dots)^4$, 4.
86. $(1+x+x^2+x^3+x^4+x^5)^4$, 140.
87. $x^3(1+x+x^2+x^3+\dots)^3$, 28.
88. $(1+x+x^2+x^3)(1+x+x^2+x^3+\dots)^3$, 164.

89. $(x^2 + x^3)(1 + x + x^2 + x^3 + \dots)^3$, 64.
90. $x^3G(x)$.
91. $G(x) - a_0 - a_1x - a_2x^2$.
92. $\frac{1}{x^3}(G(x) - a_0 - a_1x - a_2x^2)$.
93. $G(x^2)$.
94. $G(3x)$.
95. $G(x^3)$.
96. $G(x) - a_0 - a_2x^2 + 5$.
97. $a_n = -\frac{3}{4} + \frac{11}{4} \cdot 5^n$.
98. $a_n = -\frac{1}{3} \cdot 5^n + \frac{4}{3} \cdot 2^n$.
99. $a_n = \frac{11}{2} \cdot 3^n - 2^{n+1} - \frac{5}{2}$.
100. 110.
101. 200.
102. 15.
103. 515.
104. 500.
105. 542.
106. 25.
107. 667.
108. $\binom{49}{19} - \binom{20}{1} \binom{42}{19} + \binom{20}{2} \binom{35}{19} - \binom{20}{3} \binom{28}{19} + \binom{20}{4} \binom{21}{19}$.
109. $\binom{49}{19} - \binom{20}{1} \binom{36}{19} + \binom{20}{2} \binom{23}{19}$.
110. (a) $\binom{11}{2}$ (b) $\binom{21}{9} - \binom{10}{1} \binom{13}{9}$.
111. $\binom{21}{9} - \binom{10}{1} \binom{17}{9} + \binom{10}{2} \binom{13}{9} - \binom{10}{3} \binom{9}{9}$.
112. (a) $\binom{74}{4}$. (b) $\binom{104}{4} - \binom{5}{1} \binom{63}{4} + \binom{5}{2} \binom{22}{4}$.
113. $\binom{12}{7}$.
114. (a) $\binom{54}{39}$. (b) $\binom{46}{39}$. (c) $\binom{44}{37}$. (d) $\binom{54}{39} - \binom{49}{39} - \binom{47}{39} + \binom{42}{39}$.
115. $24! \cdot 3$.
116. $3 \cdot 23! - 3 \cdot 20! + 17!$.
117. $3 \cdot 22! - 18!$.
118. $26! - 3 \cdot 23! + 20!$.
119. $24!$.
120. $26! - 24!$.
121. $D_{10} = 10! - \binom{10}{1}9! + \binom{10}{2}8! - \binom{10}{3}7! + \dots + \binom{10}{10}0!$.
122. $4^8 - \binom{4}{1}3^8 + \binom{4}{2}2^8 - \binom{4}{3}1^8$.
123. $4^9 - \binom{4}{1}3^9 + \binom{4}{2}2^9 - \binom{4}{3}1^9$.
124. $5^{12} - \binom{5}{1}4^{12} + \binom{5}{2}3^{12} - \binom{5}{3}2^{12} + \binom{5}{4}1^{12}$.
125. $5^8 - \binom{5}{1}4^8 + \binom{5}{2}3^8 - \binom{5}{3}2^8 + \binom{5}{4}1^8$.
126. $a_n = a_{n-1} + a_{n-2} + 2^{n-2}$, $a_1 = 0$, $a_2 = 1$. Hence $a_8 = 201$.
127. $a_n = a_{n-1} + a_{n-3}$, $a_1 = a_2 = 1$, $a_3 = 2$. Hence $a_{12} = 60$.
128. $a_n = 2a_{n-1} + 2a_{n-2}$, $a_1 = 3$, $a_2 = 8$. Hence $a_6 = 448$.

Questions for Chapter 9

1. List all the binary relations on the set $\{0, 1\}$.
2. List the reflexive relations on the set $\{0, 1\}$.
3. List the irreflexive relations on the set $\{0, 1\}$.
4. List the symmetric relations on the set $\{0, 1\}$.
5. List the transitive relations on the set $\{0, 1\}$.
6. List the antisymmetric relations on the set $\{0, 1\}$.
7. List the asymmetric relations on the set $\{0, 1\}$.
8. List the relations on the set $\{0, 1\}$ that are reflexive and symmetric.
9. List the relations on the set $\{0, 1\}$ that are neither reflexive nor irreflexive.

In questions 10–23 determine whether the binary relation is:

(1) reflexive, (2) symmetric, (3) antisymmetric, (4) transitive.

10. The relation R on $\{1, 2, 3, \dots\}$ where aRb means $a|b$.
11. The relation R on $\{w, x, y, z\}$ where $R = \{(w, w), (w, x), (x, w), (x, x), (x, z), (y, y), (z, y), (z, z)\}$.
12. The relation R on \mathcal{Z} where aRb means $|a - b| \leq 1$.
13. The relation R on \mathcal{Z} where aRb means $a^2 = b^2$.
14. The relation R on $\{a, b, c\}$ where $R = \{(a, a), (b, b), (c, c), (a, b), (a, c), (c, b)\}$.
15. The relation R on $A = \{x, y, z\}$ where $R = \{(x, x), (y, z), (z, y)\}$.
16. The relation R on \mathcal{Z} where aRb means $a \neq b$.
17. The relation R on \mathcal{Z} where aRb means that the units digit of a is equal to the units digit of b .
18. The relation R on \mathcal{N} where aRb means that a has the same number of digits as b .
19. The relation R on the set of all subsets of $\{1, 2, 3, 4\}$ where SRT means $S \subseteq T$.
20. The relation R on the set of all people where aRb means that a is at least as tall as b .
21. The relation R on the set of all people where aRb means that a is younger than b .
22. The relation R on the set $\{(a, b) \mid a, b \in \mathcal{Z}\}$ where $(a, b)R(c, d)$ means $a = c$ or $b = d$.
23. The relation R on \mathcal{R} where aRb means $a - b \in \mathcal{Z}$.
24. A company makes four kinds of products. Each product has a size code, a weight code, and a shape code. The following table shows these codes:

	Size Code	Weight Code	Shape Code
#1	42	27	42
#2	27	38	13
#3	13	12	27
#4	42	38	38

Find which of the three codes is a primary key. If none of the three codes is a primary key, explain why.

25. If $X = (\text{Fran Williams}, 617885197, \text{MTH 202}, 248\text{B West})$, find the projections $P_{1,3}(X)$ and $P_{1,2,4}(X)$.

In questions 26–31 suppose R and S are relations on $\{a, b, c, d\}$, where

$$R = \{(a, b), (a, d), (b, c), (c, c), (d, a)\} \text{ and } S = \{(a, c), (b, d), (d, a)\}.$$

Find the combination of relations.

26. R^2 .

27. R^3 .

28. S^2 .

29. S^3 .

30. $R \circ S$.

31. $S \circ R$.

In questions 32–41 find the matrix that represents the given relation. Use elements in the order given to determine rows and columns of the matrix.

32. R on $\{1, 2, 3, 4\}$ where aRb means $|a - b| \leq 1$.

33. R on $\{w, x, y, z\}$ where $R = \{(w, w), (w, x), (x, w), (x, x), (x, z), (y, y), (z, y), (z, z)\}$.

34. R on $\{-2, -1, 0, 1, 2, \}$ where aRb means $a^2 = b^2$.

35. R on $\{1, 2, 3, 4, 6, 12\}$ where aRb means $a|b$.

36. R on $\{1, 2, 4, 8, 16\}$ where aRb means $a|b$.

37. R on $\{1, 2, 4, 8, 16\}$ where aRb means $a \leq b$.

38. R^2 , where R is the relation on on $\{1, 2, 3, 4\}$ such that aRb means $|a - b| \leq 1$.

39. R^2 , where R is the relation on $\{w, x, y, z\}$ such that $R = \{(w, w), (w, x), (x, w), (x, x), (x, z), (y, y), (z, y), (z, z)\}$.

40. R^{-1} , where R is the relation on $\{1, 2, 3, 4\}$ such that aRb means $|a - b| \leq 1$.

41. \bar{R} , where R is the relation on $\{w, x, y, z\}$ such that $R = \{(w, w), (w, x), (x, w), (x, x), (x, z), (y, y), (z, y), (z, z)\}$.

42. If $M_R = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$, determine if R is: (a) reflexive (b) symmetric (c) antisymmetric (d) transitive.

43. If $M_R = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, determine if R is: (a) reflexive (b) symmetric (c) antisymmetric (d) transitive.

44. Draw the directed graph for the relation defined by the matrix $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$,

45. Draw the directed graph for the relation defined by the matrix $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

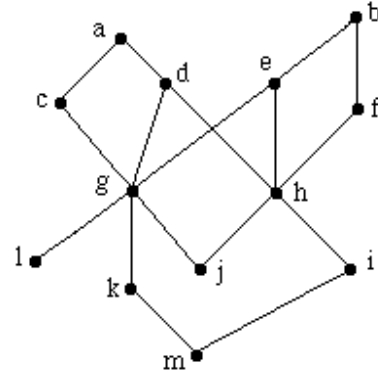
46. Draw the Hasse diagram for the relation R on $A = \{2, 3, 4, 6, 10, 12, 16\}$ where aRb means $a|b$.

47. Draw the Hasse diagram for the relation R on $A = \{2, 3, 4, 5, 6, 8, 10, 40\}$ where aRb means $a|b$.

48. Suppose $A = \{2, 3, 6, 9, 10, 12, 14, 18, 20\}$ and R is the partial order relation defined on A where xRy means x is a divisor of y .
- (a) Draw the Hasse diagram for R .
 (b) Find all maximal elements.
 (c) Find all minimal elements.
 (d) Find $\text{lub}(\{2, 9\})$.
 (e) Find $\text{lub}(\{3, 10\})$.
 (f) Find $\text{glb}(\{14, 10\})$.

49. The diagram at the right is the Hasse diagram for a partially ordered set. Referring to this diagram:

- (a) List the maximal elements
 (b) List the minimal elements
 (c) Find all upper bounds for f, g
 (d) Find all lower bounds for d, f
 (e) Find $\text{lub}(\{g, j, m\})$
 (f) Find $\text{glb}(\{d, e\})$
 (g) Find the greatest element
 (h) Find the least element
 (i) Use a topological sort to order the elements of the poset represented by this Hasse diagram.



50. Find the transitive closure of R if \mathbf{M}_R is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$.

51. Find the transitive closure of R if \mathbf{M}_R is $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$.

52. If $R = \{(1, 2), (1, 4), (2, 3), (3, 1), (4, 2)\}$, find the reflexive closure of R .
53. If $R = \{(1, 2), (1, 4), (2, 3), (3, 1), (4, 2)\}$, find the symmetric closure of R .
54. If $R = \{(x, y) \mid x \text{ and } y \text{ are bit strings containing the same number of 0s}\}$, find the equivalence classes of
 (a) 1.
 (b) 00.
 (c) 101.
55. Find the smallest equivalence relation on $\{1, 2, 3\}$ that contains $(1, 2)$ and $(2, 3)$.
56. Find the smallest partial order relation on $\{1, 2, 3\}$ that contains $(1, 1)$, $(3, 2)$, $(1, 3)$.
57. What is the covering relation of the partial ordering $\{(a, b) \mid a \text{ divides } b\}$ on the set $\{1, 2, 3, 4, 6, 8, 12, 24\}$?
58. What is the covering relation of the partial ordering $\{(a, b) \mid a \text{ divides } b\}$ on the set $\{2, 4, 6, 8, 10, 12\}$?
59. Find the join of the 3-ary relation
 $\{(Wages, MS410, N507), (Rosen, CS540, N525), (Michaels, CS518, N504), (Michaels, MS410, N510)\}$
 and the 4-ary relation
 $\{(MS410, N507, Monday, 6:00), (MS410, N507, Wednesday, 6:00), (CS540, N525, Monday, 7:30), (CS518, N504, Tuesday, 6:00), (CS518, N504, Thursday, 6:00)\}$
 with respect to the last two fields of the first relation and the first two fields of the second relation.
60. Find the transitive closure of R on $\{a, b, c, d\}$ where $R = \{(a, a), (b, a), (b, c), (c, a), (c, c), (c, d), (d, a), (d, c)\}$.

61. Which of the following are partitions of $\{1, 2, 3, \dots, 10\}$?
- (a) $\{2, 4, 6, 8\}, \{1, 3, 5, 9\}, \{7, 10\}$. (b) $\{1, 2, 4, 8\}, \{2, 5, 7, 10\}, \{3, 6, 9\}$.
 (c) $\{3, 8, 10\}, \{1, 2, 5, 9\}, \{4, 7, 8\}$. (d) $\{1\}, \{2\}, \dots, \{10\}$.
 (e) $\{1, 2, \dots, 10\}$.
62. Suppose R is the relation on \mathcal{N} where aRb means that a ends in the same digit in which b ends. Determine whether R is an equivalence relation on \mathcal{N} .
63. Suppose the relation R is defined on the set \mathcal{Z} where aRb means that $ab \leq 0$. Determine whether R is an equivalence relation on \mathcal{Z} .
64. Suppose A is the set composed of all ordered pairs of positive integers. Let R be the relation defined on A where $(a, b)R(c, d)$ means that $a + d = b + c$.
- (a) Prove that R is an equivalence relation.
 (b) Find $[(2, 4)]$.
65. Suppose that R and S are equivalence relations on a set A . Prove that the relation $R \cap S$ is also an equivalence relation on A .
66. Let R be the relation on $A = \{1, 2, 3, 4, 5\}$ where $R = \{(1, 1), (1, 3), (1, 4), (2, 2), (3, 1), (3, 3), (3, 4), (4, 1), (4, 3), (4, 4), (5, 5)\}$. Write the matrix for R .
67. Let R be the relation on $A = \{1, 2, 3, 4, 5\}$ where $R = \{(1, 1), (1, 3), (1, 4), (2, 2), (3, 1), (3, 3), (3, 4), (4, 1), (4, 3), (4, 4), (5, 5)\}$. Draw the directed graph for R .
68. Let R be the relation on $A = \{1, 2, 3, 4, 5\}$ where $R = \{(1, 1), (1, 3), (1, 4), (2, 2), (3, 1), (3, 3), (3, 4), (4, 1), (4, 3), (4, 4), (5, 5)\}$. R is an equivalence relation. Find the equivalence classes for the partition of A given by R .
- In questions 69–71 give an example or else prove that there are none.
69. A relation on $\{a, b, c\}$ that is reflexive and transitive, but not antisymmetric.
70. A relation on $\{1, 2\}$ that is symmetric and transitive, but not reflexive.
71. A relation on $\{1, 2, 3\}$ that is reflexive and transitive, but not symmetric.
72. Suppose $|A| = n$. Find the number of binary relations on A .
73. Suppose $|A| = n$. Find the number of symmetric binary relations on A .
74. Suppose $|A| = n$. Find the number of reflexive, symmetric binary relations on A .

Answers for Chapter 9

- There are 16 binary relations:

(a) $\{ \}$;	(b) $\{(0, 0)\}$;	(c) $\{(0, 1)\}$;	(d) $\{(1, 0)\}$;
(e) $\{(1, 1)\}$;	(f) $\{(0, 0), (0, 1)\}$;	(g) $\{(0, 0), (1, 0)\}$;	(h) $\{(0, 0), (1, 1)\}$;
(i) $\{(0, 1), (1, 0)\}$;	(j) $\{(0, 1), (1, 1)\}$;	(k) $\{(1, 0), (1, 1)\}$;	(l) $\{(0, 0), (0, 1), (1, 0)\}$;
(m) $\{(0, 0), (0, 1), (1, 1)\}$;	(n) $\{(0, 0), (1, 0), (1, 1)\}$;	(o) $\{(0, 1), (1, 0), (1, 1)\}$;	(p) $\{(0, 0), (0, 1), (1, 0), (1, 1)\}$.
- h, m, n, p (using the letter names in the previous question).
- a, c, d, i.
- a, b, e, h, i, l, o, p.
- All except i, l, o.
- All except i, l, o, p.
- a, c, d.

8. h, p.
9. b, e, f, g, j, k, l, o.
10. 1, 3, 4.
11. 1.
12. 1, 2.
13. 1, 2, 4.
14. 1, 3, 4.
15. 2.
16. 2.
17. 1, 2, 4.
18. 1, 2, 4.
19. 1, 3, 4.
20. 1, 4.
21. 3, 4.
22. 1, 2.
23. 1, 2, 4.
24. Shape code.
25. $P_{1,3}(X) = (\text{Fran Williams, MTH 202})$ $P_{1,2,4}(X) = (\text{Fran Williams, 617885197, 248B West})$.
26. $\{(a, a), (a, c), (b, c), (c, c), (d, b), (d, d)\}$.
27. $\{(a, b), (a, c), (a, d), (b, c), (c, c), (d, a), (d, c)\}$.
28. $\{(b, a), (d, c)\}$.
29. $\{(b, c)\}$.
30. $\{(a, c), (b, a), (d, b), (d, d)\}$.
31. $\{(a, a), (a, d), (d, c)\}$.
32.
$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$
33.
$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$
34.
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$
35.
$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

36.
$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

37.
$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

38.
$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

39.
$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

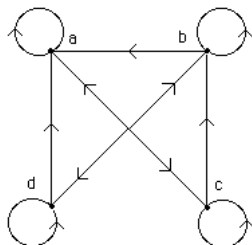
40.
$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

41.
$$\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

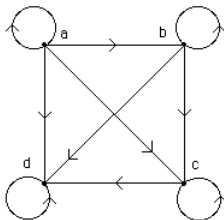
42. (a) Yes. (b) No. (c) No. (d) No.

43. (a) Yes. (b) No. (c) Yes. (d) Yes.

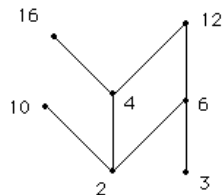
44.



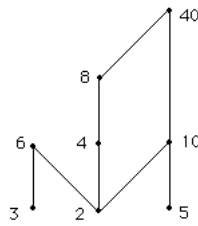
45.



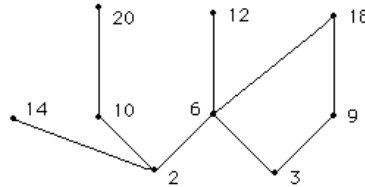
46.



47.



48. (a)



(b) 12, 14, 18, 20. (c) 2, 3. (d) 18. (e) Does not exist. (f) 2.

49. (a) a, b . (b) l, m . (c) b . (d) h, i, j, m . (e) g . (f) None. (g) None. (h) None.
 (i) For example: $m, k, i, j, l, h, g, f, e, c, d, b, a$.

50.
$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

51.
$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

52. $\{(1, 1), (1, 2), (1, 4), (2, 2), (2, 3), (3, 1), (3, 3), (4, 2), (4, 4)\}$.53. $\{(1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (4, 1), (4, 2)\}$.

54. (a) All strings that contain no 0's (including the empty string). (b) All strings with exactly two 0's. (c) All strings with exactly one 0.

55. $\{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$.56. $\{(1, 1), (2, 2), (3, 3), (3, 2), (1, 3), (1, 2)\}$.57. $\{(1, 2), (1, 3), (2, 4), (2, 6), (3, 6), (4, 8), (4, 12), (6, 12), (8, 24), (12, 24)\}$.58. $\{(2, 4), (2, 6), (2, 10), (4, 8), (4, 12), (6, 12)\}$.59. $\{(Wages, MS410, N507, Monday, 6:00), (Wages, MS410, N507, Wednesday, 6:00), (Rosen, CS540, N525, Monday, 7:30), (Michaels, CS518, N504, Tuesday, 6:00), (Michaels, CS518, N504, Thursday, 6:00)\}$.60. $\{(a, a), (b, a), (b, c), (b, d), (c, a), (c, c), (c, d), (d, a), (d, c), (d, d)\}$.

61. a, d, e.

62. Yes.

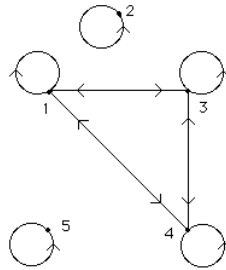
63. No (not reflexive, not transitive).

64. (a) Reflexive: $a + b = b + a$; Symmetric: if $a + d = b + c$, then $c + b = d + a$; Transitive: if $a + d = b + c$ and $c + f = d + e$, then $a + d - (d + e) = (b + c) - (c + f)$, therefore $a - e = b - f$, or $a + f = b + e$.
 (b) $[(2, 4)] = \{(a, b) \mid b = a + 2\}$.

65. Reflexive: for all $a \in A$, aRa and aSa ; hence for all $a \in A$, $a(R \cap S)a$. Symmetric: suppose $a(R \cap S)b$; then aRb and aSb ; by symmetry of R and S , bRa and bSa ; therefore $b(R \cap S)a$. Transitive: suppose $a(R \cap S)b$ and $b(R \cap S)c$; then aRb , aSb , bRc , and bSc ; by transitivity of R and S , aRc and aSc ; therefore $a(R \cap S)c$.

66.
$$\begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

67.



68. $\{1, 3, 4\}, \{2\}, \{5\}.$

69. $\{(a, a), (b, b), (c, c), (a, b), (b, a)\}.$

70. $\{(1, 1)\}.$

71. $\{(1, 1), (2, 2), (3, 3), (1, 2)\}.$

72. $2^{n^2}.$

73. $2^{n(n+1)/2}.$

74. $2^{n(n-1)/2}.$

Questions for Chapter 10

1. Construct a call graph for five friends Alice, Bob, Charlie, Diane and Evan, if there were three calls from Alice to Bob, two calls from Alice to Diane, five calls from Alice to Evan, one call from Bob to Alice, three calls from Charlie to Alice, one call from Charlie to Evan, one call from Diane to Charlie, and one call from Evan to Diane.
2. Explain how graphs can be used to model the spread of a contagious disease. Should the edges be directed or undirected? Should multiple edges be allowed? Should loops be allowed?
3. In the group stage of the 2011 women's soccer world cup the USA beat North Korea, Sweden beat Columbia, the USA beat Columbia, Sweden beat North Korea, Sweden beat the USA, and the game between Columbia and North Korea ended in a tie. Model this outcome using a directed segment from A to B if A beat B, and an undirected segment if the game ended in a tie.
4. During the construction of a home there are certain tasks that have to be completed before another one can commence, e.g., the roof has to be installed before the work on electrical wiring or plumbing can begin. How can a graph be used to model the different tasks during the construction? Should the edges be directed or undirected? Looking at the graph model, how can we find tasks that can be done at any time and how can we find tasks that do not have to be completed before other tasks can begin?
5. Many supermarkets use loyalty or discount cards to keep track of who buys which items. How can graphs be used to model this relationship? Should the edges be directed or undirected? Should multiple edges be allowed? Should loops be allowed? Does this graph have any special properties?

In questions 6–10 for each graph give an ordered pair description (vertex set and edge set) and an adjacency matrix, and draw a picture of the graph.

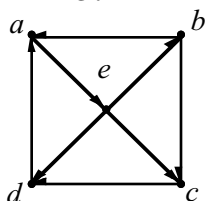
6. $K_6.$

7. C_4 .
8. W_5 .
9. $K_{4,5}$.
10. Q_3 .

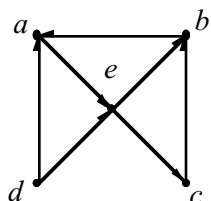
In questions 11–51 fill in the blanks.

11. K_n has _____ edges and _____ vertices.
12. $K_{m,n}$ has _____ edges and _____ vertices.
13. W_n has _____ edges and _____ vertices.
14. Q_n has _____ edges and _____ vertices.
15. The length of the longest simple circuit in K_5 is _____.
16. The length of the longest simple circuit in W_{10} is _____.
17. The length of the longest simple circuit in $K_{4,10}$ is _____.
18. List all positive integers n such that C_n is bipartite _____.
19. The adjacency matrix for $K_{m,n}$ has _____ columns.
20. The adjacency matrix for K_n has _____ 1's and _____ 0's.
21. There are _____ 0's and _____ 1's in the adjacency matrix for C_n .
22. The adjacency matrix for Q_4 has _____ entries.
23. The incidence matrix for W_n has _____ rows and _____ columns.
24. The incidence matrix for Q_5 has _____ rows and _____ columns.
25. There are _____ non-isomorphic simple undirected graphs with 5 vertices and 3 edges.
26. There are _____ non-isomorphic simple digraphs with 3 vertices and 2 edges.
27. There are _____ non-isomorphic simple graphs with 3 vertices.
28. List all positive integers n such that K_n has an Euler circuit. _____
29. List all positive integers n such that Q_n has an Euler circuit. _____
30. List all positive integers n such that W_n has an Euler circuit. _____
31. Every Euler circuit for K_9 has length _____.
32. List all positive integers n such that K_n has a Hamilton circuit. _____
33. List all positive integers n such that W_n has a Hamilton circuit. _____
34. List all positive integers n such that Q_n has a Hamilton circuit. _____
35. List all positive integers m and n such that $K_{m,n}$ has a Hamilton circuit. _____
36. Every Hamilton circuit for W_n has length _____.
37. List all positive integers n such that K_n has a Hamilton circuit but no Euler circuit. _____
38. List all positive integers m and n such that $K_{m,n}$ has a Hamilton path but no Hamilton circuit. _____
39. The largest value of n for which K_n is planar is _____.
40. The largest value of n for which $K_{6,n}$ is planar is _____.

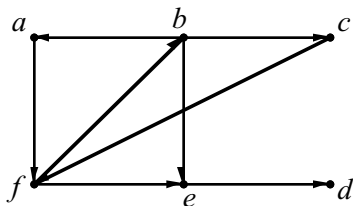
41. List all the positive integers n such that $K_{2,n}$ is planar. _____
42. The Euler formula for planar connected graphs states that _____.
43. If G is a connected graph with 12 regions and 20 edges, then G has _____ vertices.
44. If G is a planar connected graph with 20 vertices, each of degree 3, then G has _____ regions.
45. If a regular graph G has 10 vertices and 45 edges, then each vertex of G has degree _____.
46. The edge-chromatic number for $K_{2,5} =$ _____.
47. The vertex-chromatic number for $K_{7,7} =$ _____.
48. The vertex-chromatic number for $C_{15} =$ _____.
49. The region-chromatic number for $W_9 =$ _____.
50. The vertex-chromatic number for $W_9 =$ _____.
51. The vertex-chromatic number for $K_n =$ _____.
52. Determine whether the graph is strongly connected, and if not, whether it is weakly connected.



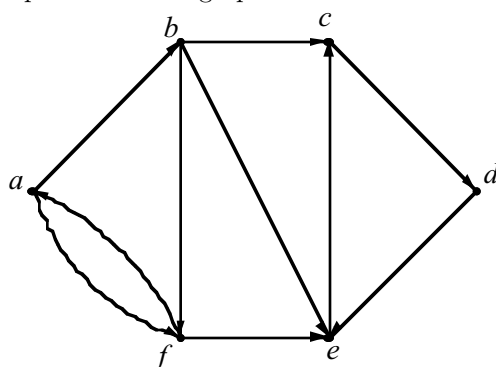
53. Determine whether the graph is strongly connected, and if not, whether it is weakly connected.



54. Find the strongly connected components of the graph.

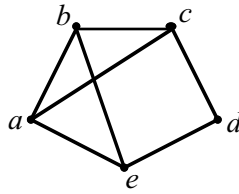


55. Find the strongly connected components of the graph.

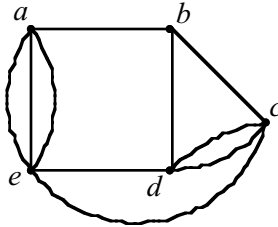


For each of the graphs in 56–58 find $\kappa(G)$, $\lambda(G)$, and $\min_{\nu \in V} \deg(\nu)$, and determine which of the two inequalities in $\kappa(G) \leq \lambda(G) \leq \min_{\nu \in V} \deg(\nu)$ are strict.

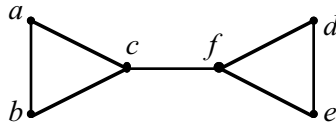
56.



57.



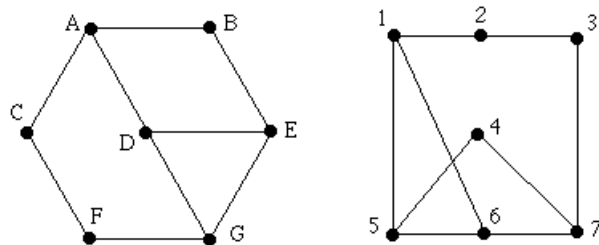
58.



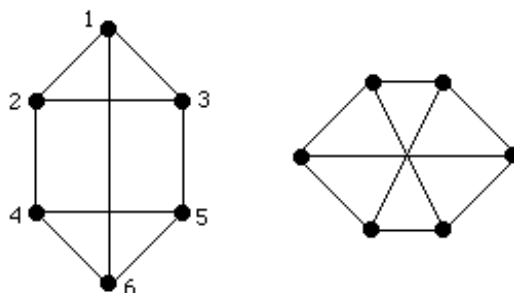
In questions 59–83 either give an example or prove that there are none.

59. A simple graph with 6 vertices, whose degrees are 2, 2, 2, 3, 4, 4.
60. A simple graph with 8 vertices, whose degrees are 0, 1, 2, 3, 4, 5, 6, 7.
61. A simple graph with degrees 1, 2, 2, 3.
62. A simple graph with degrees 2, 3, 4, 4, 4.
63. A simple graph with degrees 1, 1, 2, 4.
64. A simple digraph with indegrees 0, 1, 2 and outdegrees 0, 1, 2.
65. A simple digraph with indegrees 1, 1, 1 and outdegrees 1, 1, 1.
66. A simple digraph with indegrees 0, 1, 2, 2 and outdegrees 0, 1, 1, 3.
67. A simple digraph with indegrees 0, 1, 2, 4, 5 and outdegrees 0, 3, 3, 3, 3.
68. A simple digraph with indegrees 0, 1, 1, 2 and outdegrees 0, 1, 1, 1.
69. A simple digraph with indegrees: 0, 1, 2, 2, 3, 4 and outdegrees: 1, 1, 2, 2, 3, 4.
70. A simple graph with 6 vertices and 16 edges.
71. A graph with 7 vertices that has a Hamilton circuit but no Euler circuit.
72. A graph with 6 vertices that has an Euler circuit but no Hamilton circuit.
73. A graph with a Hamilton path but no Hamilton circuit.
74. A graph with a Hamilton circuit but no Hamilton path.
75. A connected simple planar graph with 5 regions and 8 vertices, each of degree 3.
76. A graph with 4 vertices that is not planar.
77. A planar graph with 10 vertices.
78. A graph with vertex-chromatic number equal to 6.

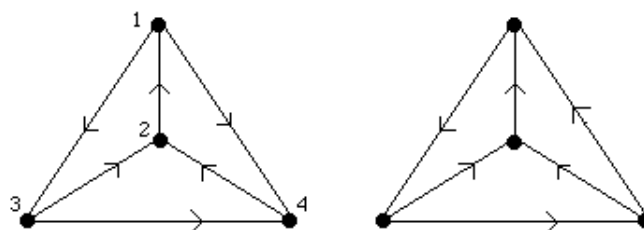
- 79. A graph with 9 vertices with edge-chromatic number equal to 2.
- 80. A graph with region-chromatic number equal to 6.
- 81. A planar graph with 8 vertices, 12 edges, and 6 regions.
- 82. A planar graph with 7 vertices, 9 edges, and 5 regions.
- 83. A bipartite graph with an odd number of vertices that has a Hamilton circuit.
- 84. Are these two graphs isomorphic?



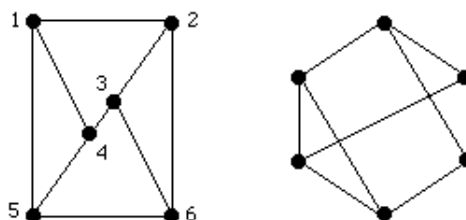
- 85. Are these two graphs isomorphic?



- 86. Are these two digraphs isomorphic?



- 87. Are these two graphs isomorphic?



88. Suppose you have a graph G with vertices v_1, v_2, \dots, v_{17} . Explain how you would use the adjacency matrix A to find
- The number of paths from v_5 to v_3 of length 12.
 - The length of a shortest path from v_5 to v_3 .
89. A simple graph is *regular* if every vertex has the same degree.
- For which positive integers n are the following graphs regular: C_n, W_n, K_n, Q_n ?
 - For which positive integers m and n is $K_{m,n}$ regular?
90. If a simple graph G has v vertices and e edges, how many edges does \overline{G} have?

91. Draw the digraph with adjacency matrix $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$.

92. Draw the undirected graph with adjacency matrix $\begin{pmatrix} 0 & 1 & 3 & 0 & 4 \\ 1 & 2 & 1 & 3 & 0 \\ 3 & 1 & 1 & 0 & 1 \\ 0 & 3 & 0 & 0 & 2 \\ 4 & 0 & 1 & 2 & 3 \end{pmatrix}$.

93. Suppose G is a graph with vertices a, b, c, d, e, f with adjacency matrix $\begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{pmatrix}$ (where alphabetical order is used to determine the rows and columns of the adjacency matrix). Find

betical order is used to determine the rows and columns of the adjacency matrix). Find

- the number of vertices in G .
- the number of edges in G .
- the degree of each vertex.
- the number of loops.
- the length of the longest simple path in G .
- the number of components in G .
- the distance between vertex a and vertex c .

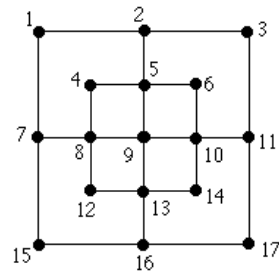
Questions 94–96 refer to a *cubic* graph, i.e., a graph that is simple and has every vertex of degree 3.

94. Draw a cubic graph with 7 vertices, or else prove that there are none.
95. Draw a cubic graph with 6 vertices that is not isomorphic to $K_{3,3}$, or else prove that there are none.
96. Draw a cubic graph with 8 edges, or else prove that there are none.
97. In K_5 find the number of paths of length 2 between every pair of vertices.
98. In K_5 find the number of paths of length 3 between every pair of vertices.
99. In K_5 find the number of paths of length 6 between every pair of vertices.
100. In $K_{3,3}$ let a and b be any two adjacent vertices. Find the number of paths between a and b of length 3.
101. In $K_{3,3}$ let a and b be any two adjacent vertices. Find the number of paths between a and b of length 4.
102. In $K_{3,3}$ let a and b be any two adjacent vertices. Find the number of paths between a and b of length 5.

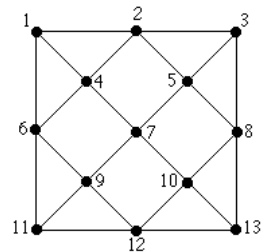
103. How many different channels are needed for six television stations (A, B, C, D, E, F) whose distances (in miles) from each other are shown in the following table? Assume that two stations cannot use the same channel when they are within 150 miles of each other?

	A	B	C	D	E	F
A	—	85	175	100	50	100
B	85	—	125	175	100	130
C	175	125	—	100	200	250
D	100	175	100	—	210	220
E	50	100	200	210	—	100
F	100	130	250	220	100	—

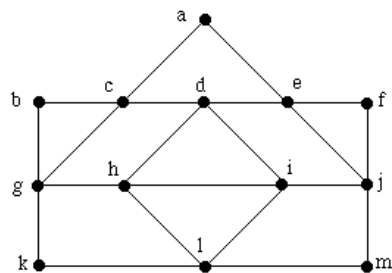
104. Consider the graph at the right.
- Does it have an Euler circuit?
 - Does it have an Euler path?
 - Does it have a Hamilton circuit?
 - Does it have a Hamilton path?



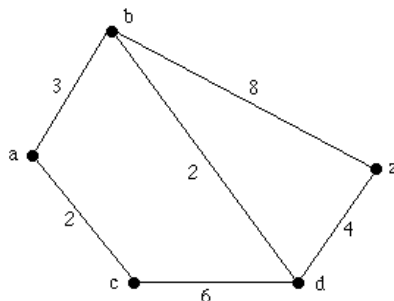
105. Consider the graph at the right.
- Does it have an Euler circuit?
 - Does it have an Euler path?
 - Does it have a Hamilton circuit?
 - Does it have a Hamilton path?



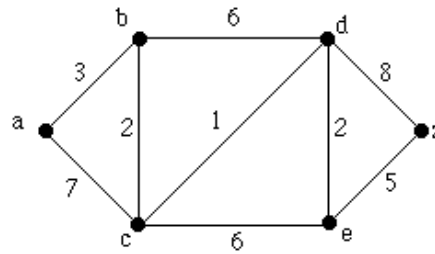
106. Consider the graph at the right.
- Does it have an Euler circuit?
 - Does it have an Euler path?
 - Does it have a Hamilton circuit?
 - Does it have a Hamilton path?



107. Use Dijkstra's Algorithm to find the shortest path length between the vertices a and z in this weighted graph.

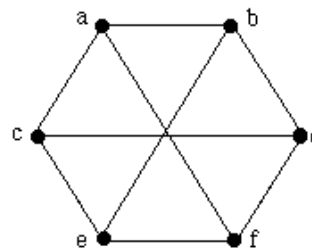


108. Use Dijkstra's Algorithm to find the shortest path length between the vertices a and z in this weighted graph.

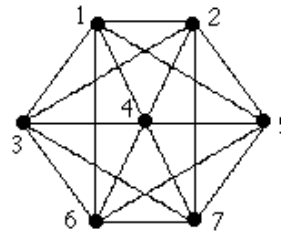


109. The Math Department has 6 committees that meet once a month. How many different meeting times must be used to guarantee that no one is scheduled to be at 2 meetings at the same time, if committees and their members are: $C_1 = \{\text{Allen, Brooks, Marg}\}$, $C_2 = \{\text{Brooks, Jones, Morton}\}$, $C_3 = \{\text{Allen, Marg, Morton}\}$, $C_4 = \{\text{Jones, Marg, Morton}\}$, $C_5 = \{\text{Allen, Brooks}\}$, $C_6 = \{\text{Brooks, Marg, Morton}\}$.

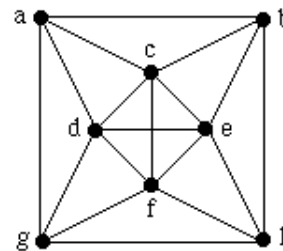
110. Determine whether this graph is planar.



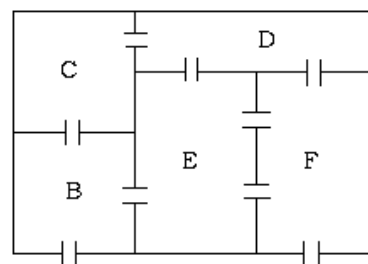
111. Determine whether this graph is planar.



112. Determine whether this graph is planar.



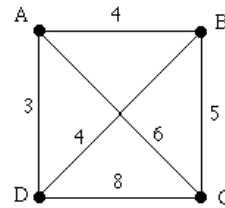
113. The picture at the right shows the floor plan of an office. Use graph theory ideas to prove that it is impossible to plan a walk that passes through each doorway exactly once, starting and ending at A.



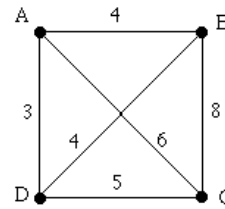
A

- 114. Find the vertex-chromatic number, the edge-chromatic number, and the region-chromatic number for $K_{3,2}$.
- 115. Find the vertex-chromatic number, the edge-chromatic number, and the region-chromatic number for K_4 .
- 116. Find the vertex-chromatic number, the edge-chromatic number, and the region-chromatic number for C_7 .
- 117. Find the vertex-chromatic number, the edge-chromatic number, and the region-chromatic number for Q_3 .
- 118. Find the vertex-chromatic number, the edge-chromatic number, and the region-chromatic number for W_5 .
- 119. Give a recurrence relation for $e_n =$ the number of edges of the graph K_n .
- 120. Give a recurrence relation for $v_n =$ number of vertices of the graph Q_n .
- 121. Give a recurrence relation for $e_n =$ number of edges of the graph Q_n .
- 122. Give a recurrence relation for $e_n =$ the number of edges of the graph W_n .

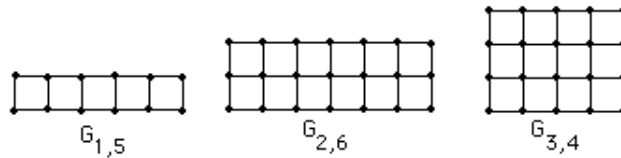
- 123. Solve the traveling salesman problem for the graph at the right by finding the total weight of all Hamilton circuits and determining a circuit with minimum total weight.



- 124. Solve the traveling salesman problem for the graph at the right by finding the total weight of all Hamilton circuits and determining a circuit with minimum total weight.



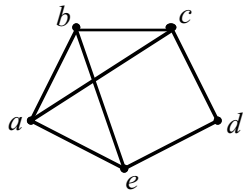
In questions 125–134 the *grid graph* $G_{m,n}$ refers to the graph obtained by taking an $m \times n$ rectangular grid of streets ($m \leq n$) with m north/south blocks and n east/west blocks. For example:



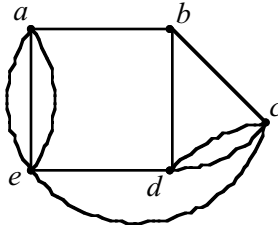
- 125. Find a formula for the number of vertices of $G_{m,n}$.
- 126. Find a formula for the number of edges of $G_{m,n}$.
- 127. Find a formula for the number of regions (including the infinite region) of $G_{m,n}$.
- 128. For which positive integers m and n does $G_{m,n}$ have an Euler circuit?
- 129. For which positive integers m and n does $G_{m,n}$ have an Euler path but no Euler circuit?
- 130. For which positive integers m and n does $G_{m,n}$ have a Hamilton circuit?
- 131. For which positive integers m and n does $G_{m,n}$ have a Hamilton path but no Hamilton circuit?
- 132. Find the vertex-chromatic number for $G_{m,n}$.
- 133. Find the edge-chromatic number for $G_{m,n}$.
- 134. Find the region-chromatic number for $G_{m,n}$ (including the infinite face).

Find the edge chromatic number for each of the graphs in 135–137.

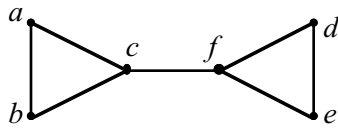
135.



136.

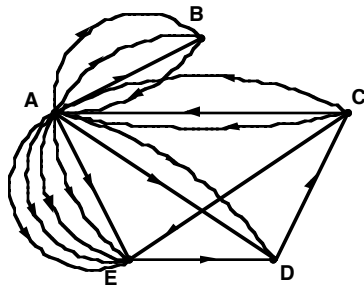


137.



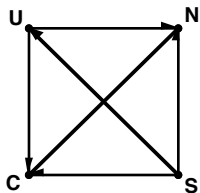
Answers for Chapter 10

1.



2. Use a directed graph, with the vertices being the individuals susceptible to the disease. An edge from u to v indicates that u infected v . Individuals with in-degree 0 are either not infected, or contracted the disease some other way (in the case of avian flu from animals), individuals with out-degree 0 contracted the disease but did not infect others before they were healed.

3.



4. Use a directed graph, with the vertices being the tasks. An edge from u to v indicates that task u has to be completed before task v can commence. Tasks that can be done at any time have an in-degree of 0, and the “finishing touches” will have out-degree 0.

5. Use vertices to represent the customers and the items stocked. An edge connecting a customer u to an item v is drawn if u buys v . Multiple edges are allowed, since customers can buy several items of the same kind.

Note that the graph is bipartite.

6. Vertices = $\{1, 2, 3, 4, 5, 6\}$; Edges = $\{\{a, b\} \mid 1 \leq a \leq 6, 1 \leq b \leq 6, a \neq b\}$;
$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

7. Vertices = $\{1, 2, 3, 4\}$; Edges = $\{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}\}$;
$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

8. Vertices = $\{1, 2, 3, 4, 5, 6\}$; Edges = $\{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{1, 6\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 6\}, \{6, 2\}\}$;

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

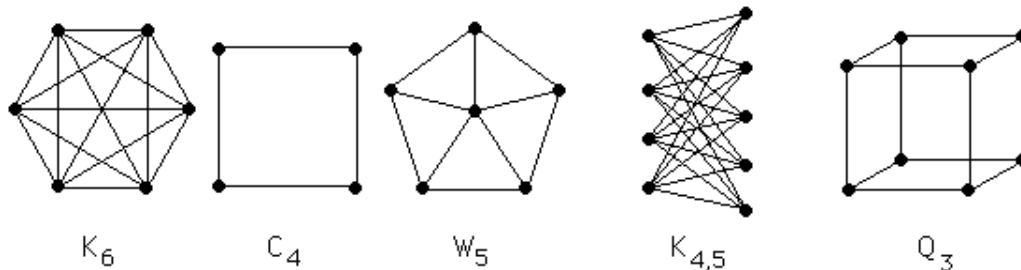
9. Vertices = $\{a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, b_5\}$, Edges = $\{\{a_i, b_j\} \mid i = 1, 2, 3, 4, j = 1, 2, 3, 4, 5\}$;

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

10. Vertices = $\{(0, 0, 0), (0, 0, 1), (0, 1, 0), (0, 1, 1), (1, 0, 0), (1, 0, 1), (1, 1, 0), (1, 1, 1)\}$;

Edges = $\{\{(a_1, a_2, a_3), (b_1, b_2, b_3)\} : |a_1 - b_1| + |a_2 - b_2| + |a_3 - b_3| = 1\}$;
$$\begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

Graphs for 6–10:



11. $n(n - 1)/2, n$.

12. $mn, m + n$.

13. $2n, n + 1$.

14. $n2^{n-1}, 2^n$.
15. 10.
16. 15.
17. 40.
18. n even.
19. $m + n$.
20. $n(n - 1), n$.
21. $n^2 - 2n, 2n$.
22. 256.
23. $n + 1, 2n$.
24. 32, 80.
25. 4.
26. 4.
27. 4.
28. n odd.
29. n even.
30. None.
31. 36.
32. All n except $n = 2$.
33. All n .
34. All n except $n = 1$.
35. $m = n > 1$.
36. $n + 1$.
37. n even ($\neq 2$).
38. $m = n + 1$ or $n = m + 1$.
39. 4.
40. 2.
41. All n .
42. $v - e + r = 2$.
43. 10.
44. 12.
45. 9.
46. 5.
47. 2.
48. 3.
49. 4 (if the infinite region is colored).
50. 4.
51. n .
52. The circuits a, e, c, d, a and a, e, b, a show that the graph is strongly connected.
53. Since there are no paths to d , the graph is not strongly connected, but is weakly connected.
54. The circuit a, f, b, c, f, b, a shows that these four vertices are in the same strong component. There are no paths from e to this strong component, and no paths from d to any other vertex. Therefore the strongly connected components are $\{a, b, c, f\}$, $\{d\}$, and $\{e\}$.

55. The circuit a, b, f, a shows that these three vertices are in the same strong component. Similarly, the circuit c, d, e, c shows that these three vertices are in the same strong component. There are no paths from $\{c, d, e\}$ to $\{a, b, f\}$. Therefore the strongly connected components are $\{a, b, f\}$, $\{c, d, e\}$.

56. $\kappa(G) = 2, \lambda(G) = 2, \min_{\nu \in V} \deg(\nu) = 2$

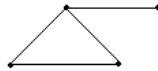
57. $\kappa(G) = 2, \lambda(G) = 4, \min_{\nu \in V} \deg(\nu) = 4$

58. $\kappa(G) = 1, \lambda(G) = 1, \min_{\nu \in V} \deg(\nu) = 2$

59. None. It is not possible to have one vertex of odd degree.

60. None. It is not possible to have a vertex of degree 7 and a vertex of degree 0 in this graph.

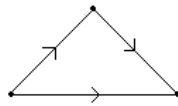
61.



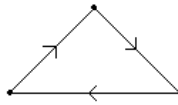
62. None. It is not possible to have a graph with one vertex of odd degree.

63. None. In a simple graph with 4 vertices, the largest degree a vertex can have is 3.

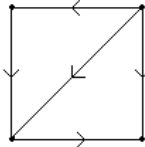
64.



65.



66.



67. None. In a simple graph with five vertices, there cannot be a vertex with indegree 5.

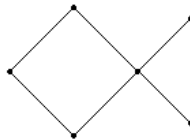
68. None. The sum of the outdegrees must equal the sum of the indegrees.

69. None. The sum of the outdegrees must equal the sum of the indegrees.

70. None. The largest number of edges in a simple graph with six vertices is 15.

71. W_6 .

72.



73. $K_{1,1}$.

74. None. Every Hamilton circuit is a Hamilton path.

75. None. The graph would have 12 edges, and hence $v - e + r = 8 - 12 + 5 = 1$, which is not possible.

76. None. The largest such graph, K_4 , is planar.

77. C_{10} .

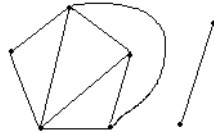
78. K_6 .

79. C_9 with one edge removed.

80. None. The 4-color theorem rules this out.

81. Q_3 .

82.



83. None. Any bipartite Hamilton graph must have an even number of vertices.

84. The graphs are isomorphic: A-7, B-4, C-3, D-6, E-5, F-2, G-1.

85. The graphs are not isomorphic: the graph on the left is planar, but the one on the right is isomorphic to $K_{3,3}$.

86. The digraphs are isomorphic: label the center vertex 4, the top vertex 2, the left vertex 1, the right vertex 3.

87. The graphs are isomorphic: label the graph clockwise from the top with 2, 3, 6, 5, 4, 1.

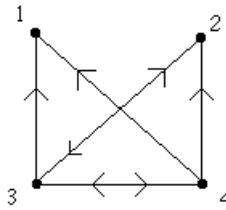
88. (a) Use the 5,3-entry of A^{12} .

(b) Examine the 5,3-entry of $A, A^2, A^3, \dots, A^{16}$. The smallest positive integer i such that the 5,3-entry of A^i is not zero is the length of a shortest path from v_5 to v_3 . If the 5,3-entry is always zero, there is no path from v_5 to v_3 .

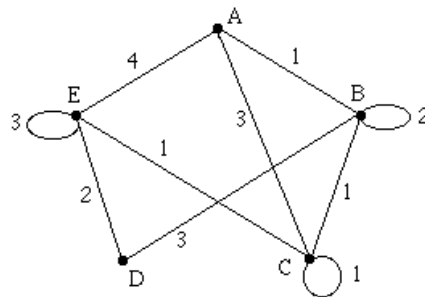
89. (a) All $n \geq 3, n = 3, \text{ all } n \geq 1, \text{ all } n \geq 0.$ (b) $m = n.$

90. $\frac{v(v-1)}{2} - e.$

91.



92. The numbers on the edges of the graph indicate the multiplicities of the edges.



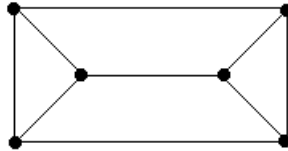
93. (a) 6.

(b) 9.

(c) 2, 4, 2, 3, 4, 3.

- (d) 0.
- (e) 9 (G has an Euler circuit).
- (f) 1.
- (g) 3.

- 94. None, since the number of vertices of odd degree must be even.
- 95. This graph is planar, whereas $K_{3,3}$ is not.



- 96. None. If $e = 8$, then $3v = 2e = 16$, which is not possible.
- 97. 3.
- 98. 13.
- 99. 819.
- 100. 9.
- 101. 0.
- 102. 81.
- 103. 4. Stations A , B , E , and F require different channels. Stations C and A can be assigned the same channel. Stations D and B can be assigned the same channel.
- 104. (a) No. (b) No. (c) No. (d) No.
- 105. (a) No. (b) No. (c) Yes. (d) Yes.
- 106. (a) Yes. (b) Yes. (c) No. (d) Yes.
- 107. First iteration: distinguished vertices a ; labels $a:0, b:3, c:2, d,z:\infty$; second iteration: distinguished vertices a, c ; labels $a:0, b:3, c:2, d:8, z:\infty$; third iteration: distinguished vertices a, b, c , labels $a:0, b:3, c:2, d:5, z:11$; fourth iteration: distinguished vertices a, b, c, d , labels $a:0, b:3, c:2, d:5, z:9$. Since z now becomes a distinguished vertex, the length of a shortest path is 9.
- 108. First iteration: distinguished vertices a ; labels $a:0, b:3, c:7, d,e,z:\infty$; second iteration: distinguished vertices a, b ; labels $a:0, b:3, c:5, d:9, e,z:\infty$; third iteration: distinguished vertices a, b, c , labels $a:0, b:3, c:5, d:6, e:11, z:\infty$; fourth iteration: distinguished vertices a, b, c, d ; labels: $a:0, b:3, c:5, d:6, e:8, z:14$; fifth iteration: distinguished vertices a, b, c, d, e ; labels $a:0, b:3, c:5, d:6, e:8, z:13$. Since z now becomes a distinguished vertex, the length of a shortest path is 13.
- 109. 5. Only C_4 and C_5 can meet at the same time.
- 110. The graph is not planar. The graph is isomorphic to $K_{3,3}$.
- 111. The graph is not planar. The graph contains a subgraph isomorphic to $K_{3,3}$, using $\{1, 3, 5\}$ and $\{2, 4, 6\}$ as the two vertex sets.
- 112. The graph is not planar. The graph contains a subgraph homeomorphic to K_5 , using vertices b, c, d, e, f .
- 113. Use vertices for rooms and edges for doorways. A walk would be an Euler circuit in this multigraph, which does not exist since B and D have odd degree.
- 114. vertex-chromatic number = 2; edge-chromatic number = 3; region-chromatic number = 3.
- 115. vertex-chromatic number = 4; edge-chromatic number = 3; region-chromatic number = 4.
- 116. vertex-chromatic number = 3; edge-chromatic number = 3; region-chromatic number = 2.
- 117. vertex-chromatic number = 2; edge-chromatic number = 3; region-chromatic number = 3.
- 118. vertex-chromatic number = 4; edge-chromatic number = 5; region-chromatic number = 4 (assuming that the infinite region is colored).
- 119. $e_n = e_{n-1} + n - 1$.

120. $v_n = 2v_{n-1}$.
121. $e_n = 2e_{n-1} + 2^{n-1}$.
122. $e_n = e_{n-1} + 2$.
123. A-D-B-C-A (weight 18).
124. A-B-D-C-A (weight 19).
125. $(m+1)(n+1)$.
126. $n(m+1) + m(n+1)$.
127. $mn + 1$.
128. $m = n = 1$.
129. $m = 1, n = 2$.
130. m or n odd.
131. m and n even.
132. 2.
133. 4.
134. 3.
135. 3
136. 6
137. 3

Questions for Chapter 11

In questions 1–26 fill in the blanks.

- If T is a tree with 999 vertices, then T has _____ edges.
- There are _____ non-isomorphic trees with four vertices.
- There are _____ non-isomorphic rooted trees with four vertices.
- There are _____ full binary trees with six vertices.
- The minimum number of weighings with a pan balance scale needed to guarantee that you find the single counterfeit coin and determine whether it is heavier or lighter than the other coins in a group of five coins is _____.
- The value of the arithmetic expression whose prefix representation is $- 5 / \cdot 6 2 - 5 3$ is _____.
- Write $3n - (k + 5)$ in prefix notation: _____.
- C_7 has _____ spanning trees.
- If each edge of Q_4 has weight 1, then the cost of any spanning tree of minimum cost is _____.
- The best comparison-based sorting algorithms for a list of n items have complexity $O(\text{_____})$.
- The bubble sort has complexity $O(\text{_____})$.
- If T is a binary tree with 100 vertices, its minimum height is _____.
- If T is a full binary tree with 101 vertices, its minimum height is _____.

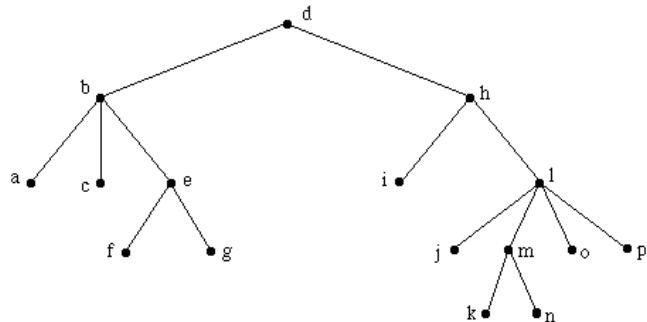
14. If T is a full binary tree with 101 vertices, its maximum height is _____.
15. If T is a full binary tree with 50 leaves, its minimum height is _____.
16. Every full binary tree with 61 vertices has _____ leaves.
17. Every full binary tree with 50 leaves has _____ vertices.
18. If T is a full binary tree of height h , then the minimum number of leaves in T is _____ and the maximum number of leaves in T is _____.
19. Every 3-ary tree with 13 vertices has _____ leaves.
20. If T is a full binary tree with 50 internal vertices, then T has _____ vertices.
21. Every full 3-ary tree of height 2 has at least _____ vertices and at most _____ vertices.
22. The largest number of leaves in a binary tree of height 5 is _____.
23. Every full binary tree with 45 vertices has _____ internal vertices.
24. A full 3-ary tree with 13 internal vertices has _____ vertices.
25. There are _____ full 3-ary trees with 6 vertices.
26. If T is a tree, then its vertex-chromatic number is _____ and its region-chromatic number is _____.

In questions 27–36 mark the statement TRUE or FALSE.

27. If T is a tree with 17 vertices, then there is a simple path in T of length 17.
28. Every tree is bipartite.
29. There is a tree with degrees 3, 2, 2, 2, 1, 1, 1, 1, 1.
30. There is a tree with degrees 3, 3, 2, 2, 1, 1, 1, 1.
31. If two trees have the same number of vertices and the same degrees, then the two trees are isomorphic.
32. If T is a tree with 50 vertices, the largest degree that any vertex can have is 49.
33. In a binary tree with 16 vertices, there must be a path of length 4.
34. Every tree is planar.
35. No tree has a Hamilton path.
36. If T is a rooted binary tree of height 5, then T has at most 25 leaves.
37. Draw all nonisomorphic trees with 5 vertices.
38. Draw all nonisomorphic rooted trees with 4 vertices.
39. Suppose T is a full m -ary tree with i internal vertices. Prove that T has $1 + (m - 1)i$ leaves.
40. Prove that if T is a full m -ary tree with l leaves, then T has $(ml - 1)/(m - 1)$ vertices.
41. Suppose T is a full m -ary tree with l leaves. Prove that T has $(l - 1)/(m - 1)$ internal vertices.
42. Prove that if T is a full m -ary tree with v vertices, then T has $((m - 1)v + 1)/m$ leaves.
43. Suppose that the universal address set address of a vertex v in an ordered rooted tree is 3.2.5.1.5. Find
 - (a) the level of v .
 - (b) the minimum number of siblings of v .
 - (c) the address of the parent of v .
 - (d) the minimum number of vertices in the tree.
44. Suppose you have 50 coins, one of which is counterfeit (either heavier or lighter than the others). You use a pan balance scale to find the bad coin. Prove that 4 weighings are not enough to guarantee that you find the bad coin and determine whether it is heavier or lighter than the other coins.

45. Suppose you have 5 coins, one of which is counterfeit (either heavier or lighter than the other four). You use a pan balance scale to find the bad coin and determine whether it is heavier or lighter.
- Prove that 2 weighings are not enough to guarantee that you find the bad coin and determine whether it is heavier or lighter.
 - Draw a decision tree for weighing the coins to determine the bad coin (and whether it is heavier or lighter) in the minimum number of weighings.
46. Suppose you have 5 coins, one of which is heavier than the other four. Draw the decision tree for using a pan balance scale to find the heavy coin.
47. (a) Set up a binary tree for the following list, in the given order, using alphabetical ordering: STOP, LET, THERE, TAPE, NONE, YOU, ANT, NINE, OAT, NUT.
- Explain step by step how you would search for the word TEST in your tree.
 - What is the height of the shortest binary search tree that can hold all 10 words?
 - Write the preorder traversal of the tree.
 - Write the postorder traversal of the tree.
 - Write the inorder traversal of the tree.
48. (a) Set up a binary tree for the following list, in the given order, using alphabetical ordering: SHE, SELLS, SEA, SHELLS, BY, THE, SEASHORE.
- How many comparisons with words in the tree are needed to determine if the word SHARK is in the tree?
 - How many comparisons with words in the tree are needed to determine if the word SEAWEED is in the tree?
 - How many comparisons with words in the tree are needed to determine if the word SHELLS is in the tree?
49. Draw a parsing tree for $(a - (3 + 2b))/(c^2 + d)$.
50. Find the preorder traversal of the parsing tree for $(8x - y)^5 - 7\sqrt{4z - 3}$.
51. Find the postorder traversal of the parsing tree for $(8x - y)^5 - 7\sqrt{4z - 3}$.
52. Find the inorder traversal of the parsing tree for $(8x - y)^5 - 7\sqrt{4z - 3}$.

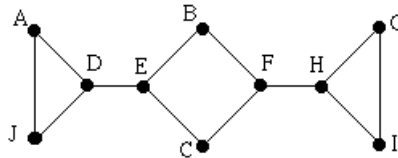
Questions 53–55 refer to the tree at the right.



- Find the preorder traversal.
 - Find the inorder traversal.
 - Find the postorder traversal.
56. The algebraic expression $\frac{a - 7c + 3}{4 + 3b}$ is written in prefix notation. Write the expression in postfix notation.
57. Write the compound proposition $(\neg p) \rightarrow (q \vee (r \wedge \neg s))$ in postfix notation.
58. Write the compound proposition $(\neg p) \rightarrow (q \vee (r \wedge \neg s))$ in prefix notation.
59. Write the compound proposition $(\neg p) \rightarrow (q \vee (r \wedge \neg s))$ in infix notation.
60. The string $2\ 3\ a \cdot x + 4\ \uparrow + 7\ \uparrow$ is postfix notation for an algebraic expression. Write the expression in prefix notation.
61. The string $2\ 3\ a \cdot x + 4\ \uparrow + 7\ \uparrow$ is postfix notation for an algebraic expression. Write the expression in infix notation.
62. The string $- \cdot 2 - x a + 4 y$ is prefix notation for an algebraic expression. Write the expression in postfix notation.

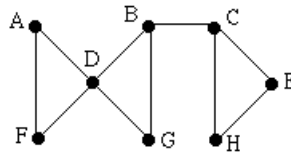
- 63. The string $- \cdot 2 - x a + 4 y$ is prefix notation for an algebraic expression. Write the expression in infix notation.
- 64. The string $p r q \rightarrow \neg q \Delta p \rightarrow \wedge$ is postfix notation for a logic expression; however, there is a misprint. The triangle should be one of these three: r , \vee , or \neg . Determine which of these three it must be and explain your reasoning.
- 65. Find the value of $- \uparrow x \cdot 5 t / 4 - 7 c$ (in prefix notation) if $c = 5$, $x = 2$, and $t = 1$.

Questions 66–73 refer to this graph.



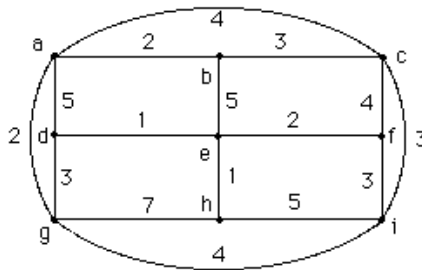
- 66. Using alphabetical ordering, find a spanning tree for this graph by using a depth-first search.
- 67. Using alphabetical ordering, find a spanning tree for this graph by using a breadth-first search.
- 68. Using the ordering C, D, E, F, G, H, I, J, A, B, C, find a spanning tree for this graph by using a depth-first search.
- 69. Using the ordering C, D, E, F, G, H, I, J, A, B, C, find a spanning tree for this graph by using a breadth-first search.
- 70. Using reverse alphabetical ordering, find a spanning tree for the graph by using a depth-first search.
- 71. Using reverse alphabetical ordering, find a spanning tree for the graph by using a breadth-first search.
- 72. Using the ordering B, G, J, A, C, I, F, H, D, E, find a spanning tree for this graph by using a depth-first search.
- 73. Using the ordering B, G, J, A, C, I, F, H, D, E, find a spanning tree for this graph by using a breadth-first search.

Questions 74–81 refer to this graph.

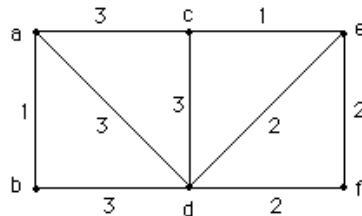


- 74. Using alphabetical ordering, find a spanning tree for this graph by using a depth-first search.
- 75. Using alphabetical ordering, find a spanning tree for this graph by using a breadth-first search.
- 76. Using the ordering C, D, E, F, G, H, I, J, A, B, C, find a spanning tree for this graph by using a depth-first search.
- 77. Using the ordering C, D, E, F, G, H, I, J, A, B, C, find a spanning tree for this graph by using a breadth-first search.
- 78. Using reverse alphabetical ordering, find a spanning tree for the graph by using a depth-first search.
- 79. Using reverse alphabetical ordering, find a spanning tree for the graph by using a breadth-first search.
- 80. Using the ordering B, G, J, A, C, I, F, H, D, E, find a spanning tree for this graph by using a depth-first search.
- 81. Using the ordering B, G, J, A, C, I, F, H, D, E, find a spanning tree for this graph by using a breadth-first search.

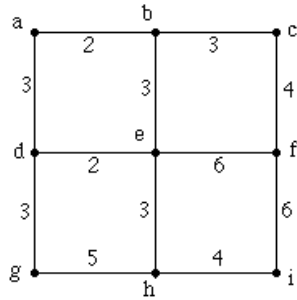
- 82. Find a spanning tree for the graph $K_{3,4}$ using a depth-first search. (Assume that the vertices are labeled u_1, u_2, u_3 in one set and v_1, v_2, v_3, v_4 in the other set, and that alphabetical ordering is used in the search, with numerical ordering on the subscripts used to break ties.)
- 83. Find a spanning tree for the graph $K_{3,4}$ using a breadth-first search. (Assume that the vertices are labeled u_1, u_2, u_3 in one set and v_1, v_2, v_3, v_4 in the other set, and that alphabetical ordering is used in the search, with numerical ordering on the subscripts used to break ties.)
- 84. Is the following code a prefix code: A: 11, B: 10, C: 0?
- 85. Is the code given by $a : 0, m : 10, s : 110, t : 111$ a prefix code?
- 86. Is the code given by $c : 00, h : 10, d : 1101, e : 101, z : 111$ a prefix code?
- 87. Use Huffman coding to encode these symbols with given frequencies: $a : 0.35, b : 0.4, c : 0.2, d : 0.05$. What is the average number of bits required to encode a character?
- 88. Use Huffman coding to encode these symbols with given frequencies: $a : 0.15, b : 0.35, c : 0.23, d : 0.22, e : 0.04, f : 0.01$. What is the average number of bits required to encode a character?
- 89. Use the bubble sort to sort the list 5, 2, 3, 1, 4 in increasing order.
- 90. Use the merge sort to sort the list 4, 8, 6, 1, 5, 7, 3, 2 in increasing order.
- 91. Use the bubble sort to sort the list 5, 4, 3, 2, 1 in increasing order.
- 92. Use the merge sort to sort the list 3, 8, 12, 4, 1, 5, 9, 6 in increasing order.
- 93. Use backtracking to find a sum of integers in the set {18, 19, 23, 25, 31} that equals 44.
- 94. Find a minimal spanning tree for this weighted graph using Prim's algorithm.



- 95. Use Prim's algorithm to find a minimal spanning tree for this weighted graph. Use alphabetical order to break ties.



96. Find a spanning tree of minimum cost for this graph.

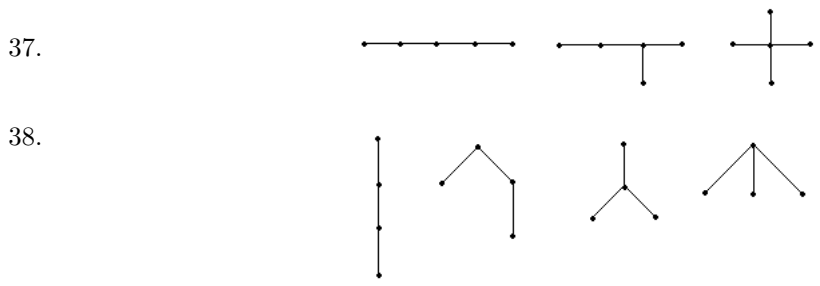


97. Describe the difference between Prim's algorithm and Kruskal's algorithm for finding a spanning tree of minimum cost.

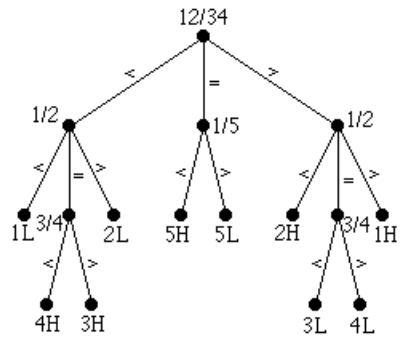
Answers for Chapter 11

1. 998.
2. 2.
3. 4.
4. 0.
5. 3.
6. -1.
7. $-3n + k5$.
8. 7.
9. 15.
10. $n \log_2 n$.
11. n^2 .
12. 6.
13. 6.
14. 50.
15. 6.
16. 31.
17. 99.
18. $h + 1, 2^h$.
19. 9.
20. 101.
21. 7, 13.
22. 32.
23. 22.
24. 40.
25. 0.
26. 2, 1.
27. False.
28. True.

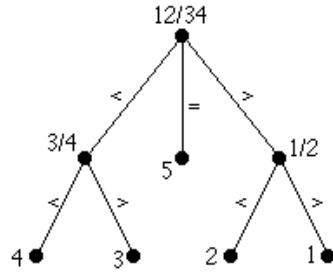
- 29. False.
- 30. True.
- 31. False.
- 32. True.
- 33. True.
- 34. True.
- 35. False.
- 36. False.



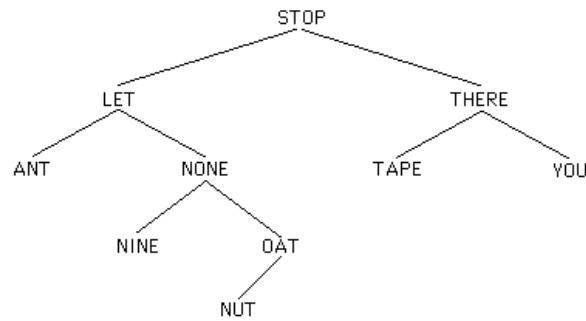
- 39. $mi + 1 = v = i + l$. Therefore $l = mi + 1 - i = 1 + (m - 1)i$.
- 40. $mi + 1 = v$ and $i = v - l$. Therefore $m(v - l) + 1 = v$. Solve for v to obtain the formula.
- 41. $i + l = v = mi + 1$. Therefore $i + l = mi + 1$. Solve for i to obtain the result.
- 42. $i + l = v$ and $mi + 1 = v$. Therefore $v - l = i$ and $i = (v - 1)/m$. Hence, $v - l = (v - 1)/m$. Solve for l .
- 43. (a) 5. (b) 4. (c) 3.2.5.1. (d) 17.
- 44. Four weighings yield a 3-ary tree of height 4, which has at most 81 leaves. Fifty coins require a tree with 100 leaves.
- 45. (a) Two weighings yield a 3-ary tree of height 2, which has at most 9 leaves, but 5 coins require a tree with 10 leaves.
 (b) Use the weighing 1 and 2 against 3 and 4 as the root. If the four coins have the same weight, weigh 1 against 5 to determine whether 5 is heavy or light. If 1 and 2 are lighter or heavier than 3 and 4, weigh 1 against 2. If 1 and 2 balance, weigh 3 against 4 to find out which of these coins is heavier or lighter; if 1 and 2 do not balance, then immediate information is obtained regarding coins 1 or 2. (The “<” symbol on an edge means that the coins in the left pan weigh less than the coins in the right pan.)



46. Weigh 1 and 2 against 3 and 4. If they balance, 5 is the bad coin. If 1 and 2 weigh less than 3 and 4, weigh 3 against 4 to find which of 3 or 4 is bad. If 1 and 2 weigh more than 3 and 4, weigh 1 against 2 to find out which of 1 or 2 is bad.



47. (a)



(b) In sequence, TEST would be compared with STOP, THERE, TAPE, and inserted as the right child of TAPE.

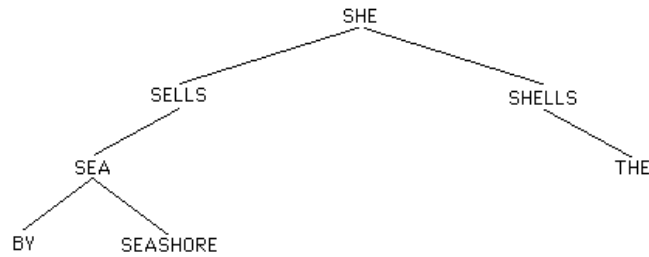
(c) 3.

(d) STOP, LET, ANT, NONE, NINE, OAT, NUT, THERE, TAPE, YOU.

(e) ANT, NINE, NUT, OAT, NONE. LET, TAPE, YOU, THERE, STOP.

(f) ANT, LET, NINE, NONE, NUT, OAT, STOP, TAPE, THERE, YOU.

48. (a)

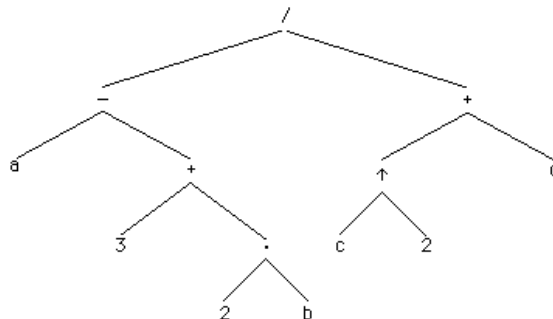


(b) 2.

(c) 4.

(d) 4.

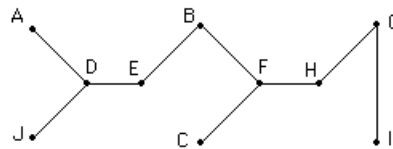
- 49.



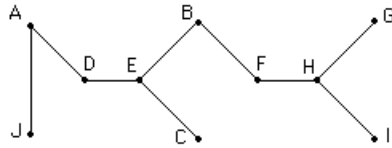
- 50. $\neg \uparrow \neg \cdot 8 x y 5 \cdot 7 \sqrt{\neg \cdot 4 z 3}$
- 51. $8 x \cdot y \neg 5 \uparrow \neg 4 z \cdot 3 \neg \sqrt{\cdot \neg}$
- 52. $8 \cdot x \neg y \uparrow 5 \neg 7 \cdot 4 \cdot z \neg 3 \sqrt{\cdot}$
- 53. *dbacefghijklmnop*
- 54. *abcfe g d i h j l k m n o p*
- 55. *a c f g e b i j k n m o p l h d*
- 56. $a 7 c \cdot \neg 3 \uparrow 4 \neg 3 b \cdot /$
- 57. $p \neg q r s \neg \wedge \vee \rightarrow$
- 58. $\rightarrow \neg p \vee q \wedge r \neg s$
- 59. $p \neg \rightarrow q \vee r \wedge s \neg$
- 60. $\uparrow + 2 \uparrow + \cdot 3 a x 4 7$
- 61. $2 + 3 \cdot a + x \uparrow 4 \uparrow 7$
- 62. $2 x a \neg \cdot 4 y + \neg$
- 63. $2 \cdot x \neg a \neg 4 + y$

- 64. The triangle should be \vee . Using either r or \neg makes the parsing tree impossible to draw.
- 65. 30.

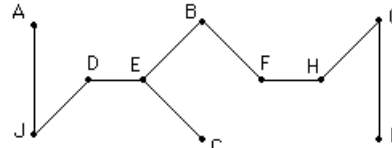
66.



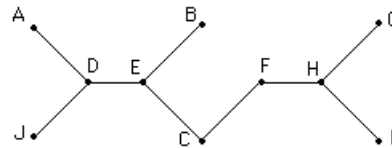
67.



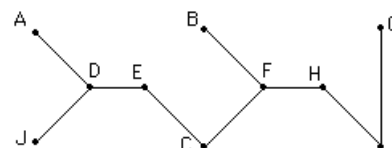
68.



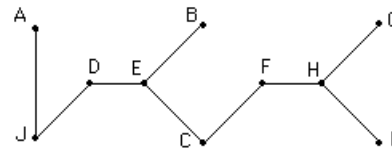
69.



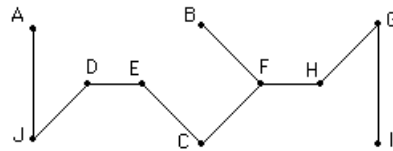
70.



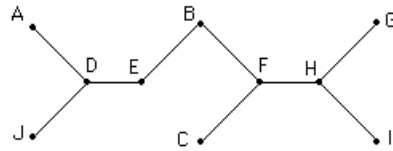
71.



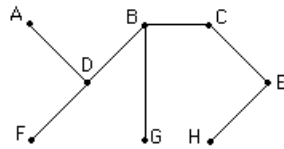
72.



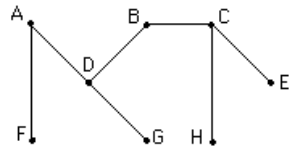
73.



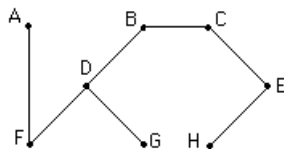
74.



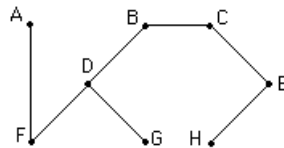
75.



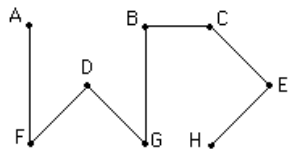
76.



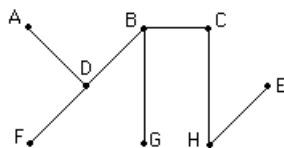
77.



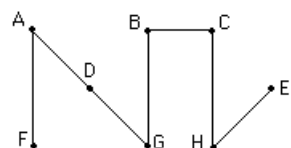
78.



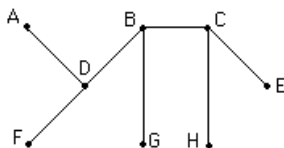
79.



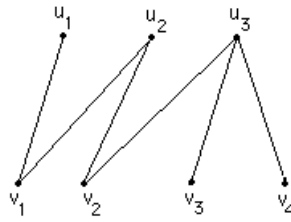
80.



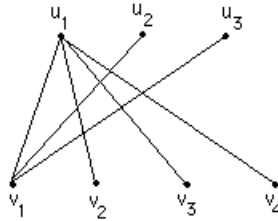
81.



82.



83.



84. Yes

85. Yes, since no code is the first part of another.

86. No, the code for h is the first part of the code for e .

87. The codes for $a, b, c,$ and d are 00, 1, 010, and 011, respectively. On average 1.85 bits are needed per character.

88. The codes for $a, b, c, d, e,$ and f are 110, 00, 01, 10, 1110 and 1111, respectively. On average 2.25 bits are needed per character.

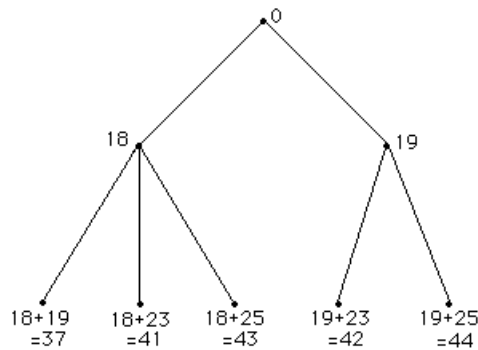
89. 2, 5, 3, 1, 4, 2, 3, 5, 1, 4, 2, 3, 1, 5, 4, 2, 3, 1, 4, 5; 2, 3, 1, 4, 5, 2, 1, 3, 4, 5, 2, 1, 3, 4, 5; 1, 2, 3, 4, 5, 1, 2, 3, 4, 5; 1, 2, 3, 4, 5.

90. After splitting the list into eight lists of 1 element each, the lists are merged into four sorted lists of 2 elements each: 4, 8; 1, 6; 5, 7; 2, 3. These are merged into two sorted lists of 4 elements: 1, 4, 6, 8 and 2, 3, 5, 7. Finally, these are merged into the final sorted list 1, 2, 3, 4, 5, 6, 7, 8.

91. 5, 4, 3, 2, 1, 4, 5, 3, 2, 1, 4, 3, 5, 2, 1, 4, 3, 2, 5, 1, 4, 3, 2, 1, 5; 3, 4, 2, 1, 5, 3, 2, 4, 1, 5, 3, 2, 1, 4, 5; 2, 3, 1, 4, 5, 2, 1, 3, 4, 5; 1, 2, 3, 4, 5.

92. 3, 8; 4, 12; 1, 5; 6, 9. 3, 4, 8, 12; 1, 5, 6, 9. 1, 3, 4, 5, 6, 8, 9, 12.

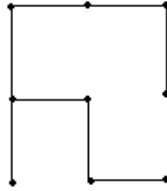
93. The following tree is obtained using backtracking; it yields $44 = 19 + 25$.



94. In order, the following edges are added: $\{d, e\}, \{e, h\}, \{e, f\}, \{d, g\}, \{g, a\}, \{a, b\}, \{b, c\}, \{c, i\}$. The weight of the minimal spanning tree is 17.

95. In order, the following edges are added: $\{a, b\}, \{a, c\}, \{c, e\}, \{d, e\}, \{d, f\}$. The weight is 9.

96. The minimum weight is 24. One example of a tree is



97. Using Prim's algorithm, at each stage the edges selected will form a tree, whereas in Kruskal's algorithm this need not happen.

Questions for Chapter 12

In questions 1–8 fill in the blanks.

1. The idempotent laws in a Boolean algebra state that _____ and _____.
2. There are _____ Boolean functions with 2 variables.
3. There are _____ Boolean functions with 3 variables.
4. There are _____ Boolean functions with 4 variables.
5. Using “ \downarrow ” for “nor”, $(x \downarrow y) \downarrow (x \downarrow y)$ can be written in terms of $\bar{}$, $+$, and \cdot as _____.
6. Using “ \downarrow ” for “nor”, $(x \downarrow x) \downarrow (y \downarrow y)$ can be written in terms of $\bar{}$, $+$, and \cdot as _____.
7. When written as a sum of minterms (in the variables x and y), $x + \bar{x}y =$ _____.
8. When written as a product of maxterms (in the variables x and y), $(x + y)z =$ _____.

In questions 9–22 mark each statement TRUE or FALSE.

9. When written as a sum of minterms in the variables x and y , $x + \bar{y} = xy + x\bar{y} + \bar{x}\bar{y}$.
10. When written as a sum of minterms in the variables x and y , $1 = xy + x\bar{y} + \bar{x}y + \bar{x}\bar{y}$.
11. If $f(z, y, z) = xyz$, then $\overline{f(z, y, z)} = \bar{x}\bar{y}\bar{z}$.
12. $xx + xy + yy = xy$.
13. Every Boolean function can be written using only the operators $\bar{}$, $+$, and \cdot .
14. There are n^2 minterms in the variables x_1, x_2, \dots, x_n .
15. $\bar{x}\bar{y} = ((x|x)|(y|y)) | ((x|x)|(y|y))$.
16. $x + y = \bar{x} | \bar{y}$.
17. $x \downarrow y = \overline{x + y}$.
18. $\overline{x \downarrow y} = \bar{x} | \bar{y}$.
19. $\overline{x | y} = x \downarrow y$.
20. $\{+, \cdot\}$ is a functionally complete set of operators.
21. The circuit diagrams for $x + \bar{x}y$ and $y + x\bar{y}$ produce the same output.

22. The circuit diagrams for $\overline{\overline{x}\overline{y} + \overline{x}y}$ and $x + y$ produce the same output.
23. Write $x + y$ as a sum-of-products in the variables x and y .
24. Write $x(y + 1)$ as a sum-of-products in the variables x and y .
25. Write $(x + y)(\overline{x} + \overline{y})$ as a sum-of-products in the variables x and y .
26. Write 1 as a sum-of-products in the variables x and y .
27. Write $x + y + z$ as a sum-of-products in the variables x , y , and z .
28. Write $xy + xy\overline{z}$ as a sum-of-products in the variables x , y , and z .
29. Write $(x + yz)(\overline{x} + yz)$ as a sum-of-products in the variables x , y , and z .
30. Write $x + z$ as a sum-of-products in the variables x , y , and z .
31. Write $x\overline{y}z$ as a sum-of-products in the variables x , y , and z .
32. Find the sum-of-products expansion of the Boolean function $f(x, y)$ that is 1 if and only if either $x = 0$ and $y = 1$, or $x = 1$ and $y = 0$.
33. Find the sum-of-products expansion of the Boolean function $f(x, y, z)$ that is 1 if and only if exactly two of the three variables have value 1.
34. Find the sum-of-products expansion of the Boolean function $f(x, y, z)$ that is 1 if and only if either $x = z = 1$ and $y = 0$, or $x = 0$ and $y = z = 1$.
35. Find a Boolean function $F : \{0, 1\}^2 \rightarrow \{0, 1\}$ such that $F(0, 0) = F(0, 1) = F(1, 1) = 1$ and $F(1, 0) = 0$.
36. (a) Find a Boolean function $f : \{0, 1\}^3 \rightarrow \{0, 1\}$ such that $f(1, 1, 0) = 1$, $f(0, 1, 1) = 1$, and $f(x, y, z) = 0$ otherwise.
(b) Write f using only \cdot and $\overline{\quad}$.
37. If $f(w, x, y, z) = \overline{(\overline{x} + y\overline{z})} + (\overline{w}x)$, find $f(1, 1, 1, 1)$.
38. If $f(w, x, y, z) = \overline{(\overline{x} + y\overline{z})} + (\overline{w}x)$, find $f(0, 1, 0, 1)$.
39. If $f(w, x, y, z) = \overline{(\overline{x} + y\overline{z})} + (\overline{w}x)$, find $f(1, 0, 1, 1)$.
40. If $f(w, x, y, z) = \overline{(\overline{x} + y\overline{z})} + (\overline{w}x)$, find $f(0, 0, 0, 0)$.
41. If $f(w, x, y, z) = \overline{(\overline{x} + y\overline{z})} + (\overline{w}x)$, find $f(1, 1, 0, 0)$.
42. If $f(w, x, y, z) = \overline{(\overline{x} + y\overline{z})} + (\overline{w}x)$, find $f(0, 0, 1, 0)$.
43. Prove that $F = G$, where $F(x, y) = \overline{(x + \overline{x}y)}\overline{y}$ and $G(x, y) = \overline{x + y}$.
44. Show that the Boolean function F given by $F(x, y, z) = x(z + yz) + y\overline{(\overline{x}z\overline{x})}$ simplifies to $xz + \overline{x}y$, by using only the definition of a Boolean algebra.
45. Show that the Boolean function F given by $F(x, y, z) = \overline{\overline{x + y}} + xy + \overline{x + y}$ simplifies to $x + \overline{y}$, by using only the definition of a Boolean algebra.
46. Using only the five properties associative laws, commutative laws, distributive laws, identity laws, and complement laws, prove that $xx = x$ is true in all Boolean algebras.
47. Using only the five properties associative laws, commutative laws, distributive laws, identity laws, and complement laws, prove that $x + x = x$ is true in all Boolean algebras.
48. Using only the five properties associative laws, commutative laws, distributive laws, identity laws, and complement laws, prove that $x + (xy) = x$ is true in all Boolean algebras.
49. Using only the five properties associative laws, commutative laws, distributive laws, identity laws, and complement laws, prove that $x + 1 = 1$ is true in all Boolean algebras.

In questions 50–61 determine whether the statement is TRUE or FALSE. Assume that x , y , and z represent Boolean variables.

50. $x + x y z = x$.

51. $x + x y + x = x$.

52. $\overline{x + y} = \bar{x} + \bar{y}$.

53. $x(x + y) = x + y x$.

54. $\bar{x} z + x z = z$.

55. $x + y + z = x y z$.

56. $x + x \bar{y} = x y z + x(\bar{z} + \bar{y} z)$.

57. $(x x + 1) = (x + 1)(x + 1)$.

58. $\overline{z + x y} = \bar{z} + \bar{x} \bar{y}$.

59. $\overline{y + x z} = \bar{y} \bar{x} + \bar{y} \bar{z}$.

60. $\overline{y z + \bar{x}} = \bar{y} \bar{z} x$.

61. $(0 + x)(1 + x) = x x$.

62. Prove that the set of real numbers, with addition and multiplication of real numbers as $+$ and \cdot , negation as complementation, and the real numbers 0 and 1 as the 0 and the 1 respectively, is not a Boolean algebra.

63. Give a reason for each step in the proof that $x + x = x$ is true in Boolean algebras. Your reasons should come from the following: associative laws for addition and multiplication, commutative laws for addition and multiplication, distributive law for multiplication over addition and distributive law for addition over multiplication, identity laws, unit property, and zero property.

$$x = x + 0 = x + (x \bar{x}) = (x + x)(x + \bar{x}) = (x + x) \cdot 1 = 1 \cdot (x + x) = x + x.$$

64. Give a reason for each step in the proof that $x + 1 = x$ is true in Boolean algebras. Your reasons should come from the following: associative laws for addition and multiplication, commutative laws for addition and multiplication, distributive law for multiplication over addition and distributive law for addition over multiplication, identity laws, unit property, and zero property.

$$1 = x + \bar{x} = x + \bar{x} \cdot 1 = (x + \bar{x})(x + 1) = 1 \cdot (x + 1) = x + 1.$$

65. Give a reason for each step in the proof that $x + x y = x$ is true in Boolean algebras. Your reasons should come from the following: associative laws for addition and multiplication, commutative laws for addition and multiplication, distributive law for multiplication over addition and distributive law for addition over multiplication, identity laws, unit property, zero property, and idempotent laws.

$$x + x y = x \cdot 1 + x y = x(y + \bar{y}) + x y = (x y + x \bar{y}) + x y = x y + (x y + x \bar{y}) = (x y + x y) + x \bar{y} = x y + x \bar{y} = x(y + \bar{y}) = x \cdot 1 = x.$$

66. Draw a logic gate diagram for the Boolean function $F(x, y, z) = \overline{(x \bar{y})} + x \bar{z}$.

67. Let $F(x, y, z) = \bar{y}(\bar{x} z) + y x + y \bar{z}$. Draw a logic gate diagram for F .

68. Let $F(x, y, z) = \bar{y}(\bar{x} z) + y x + y \bar{z}$. Use a Karnaugh map to simplify the function F .

69. Use a Karnaugh map to minimize the sum-of-products expression $x y z + x \bar{y} z + x \bar{y} \bar{z} + \bar{x} \bar{y} z$.

70. Use a Karnaugh map to minimize the sum-of-products expression $x y z + x \bar{y} z + \bar{x} \bar{y} z + x \bar{y} \bar{z} + \bar{x} y z + \bar{x} \bar{y} \bar{z}$.

71. Construct a circuit using inverters, OR gates, and AND gates that gives an output of 1 if and only if three people on a committee do not all vote the same.

72. Let $F(x, y, z) = \overline{(\bar{y} z)}(x + \bar{x} y)$. Draw a logic gate diagram for F .

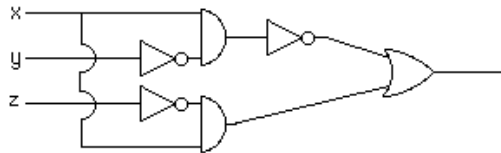
73. Let $F(x, y, z) = \overline{(\overline{y}z)}(x + \overline{x}y)$. Show that F can be simplified to give $y + x\overline{z}$.
74. A circuit is to be built that takes the numbers 0 through 9 as inputs ($1 = 0001, 2 = 0010, \dots, 9 = 1001$). Let $F(w, x, y, z)$ be the Boolean function that produces an output of 1 if and only if the input is an even number. Find a Karnaugh map for F and use the map and don't care conditions to find a simple expression for F .
75. A circuit is to be built that takes the numbers 0 through 9 as inputs ($1 = 0001, 2 = 0010, \dots, 9 = 1001$). Let $G(w, x, y, z)$ be the Boolean function that produces an output of 1 if and only if the input is an odd number. Find a Karnaugh map for G and use the map and don't care conditions to find a simple expression for G .
76. Use the Quine-McCluskey method to simplify the Boolean expression $\overline{x}yz + \overline{x}y\overline{z} + \overline{x}\overline{y}z + \overline{x}\overline{y}\overline{z} + xyz$.
77. Use the Quine-McCluskey method to simplify the Boolean expression $wxyz + wxy\overline{z} + w\overline{x}yz + w\overline{x}y\overline{z} + \overline{w}\overline{x}yz + w\overline{x}\overline{y}z + \overline{w}\overline{x}\overline{y}z$

Answers for Chapter 12

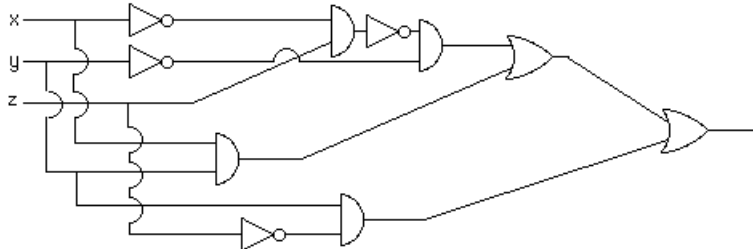
1. $x + x = x, x \cdot x = x$.
2. 2^4 .
3. 2^8 .
4. 2^{16} .
5. $x + y$.
6. xy .
7. $xy + x\overline{y} + \overline{x}y$.
8. $(x + y + z)(x + \overline{y} + z)(x + y + \overline{z})(\overline{x} + y + z)(\overline{x} + \overline{y} + z)$.
9. True.
10. True.
11. False.
12. False.
13. True.
14. False.
15. True.
16. True.
17. True.
18. True.
19. False.
20. False.
21. True.
22. False.
23. $xy + x\overline{y} + \overline{x}y$.
24. $xy + x\overline{y}$.
25. $x\overline{y} + \overline{x}y$.
26. $xy + \overline{x}y + x\overline{y} + \overline{x}\overline{y}$.
27. $xyz + \overline{x}yz + x\overline{y}z + xy\overline{z} + \overline{x}\overline{y}z + \overline{x}y\overline{z} + x\overline{y}\overline{z}$.
28. $xyz + xy\overline{z}$.

29. $x y z + \bar{x} y z.$
30. $x y z + x \bar{y} z + x y \bar{z} + x \bar{y} \bar{z} + \bar{x} y z + \bar{x} \bar{y} z.$
31. $x \bar{y} z.$
32. $\bar{x} y + x \bar{y}.$
33. $\bar{x} y z + x \bar{y} z + x y \bar{z}.$
34. $x \bar{y} z + \bar{x} y z.$
35. $\overline{x \bar{y}}.$
36. (a) $x y \bar{z} + \bar{x} y z$ (b) $\overline{x y \bar{z} \cdot \bar{x} y z}.$
37. 1.
38. 1.
39. 0.
40. 0.
41. 0.
42. 0.
43. $F = \overline{x + \bar{x} y} \cdot \bar{y} = \bar{x} \cdot \overline{\bar{x} y} \cdot \bar{y} = \bar{x} (x + \bar{y}) \bar{y} = (\bar{x} x + \bar{x} \bar{y}) \bar{y} = \bar{x} x \bar{y} + \bar{x} \bar{y} \bar{y} = 0 + \bar{x} \bar{y} = \bar{x} \bar{y} = \overline{x + y} = G.$
44. $x (z + y z) + y \cdot \overline{\bar{x} \bar{z} \cdot \bar{x}} = x z + x y z + y (\overline{\bar{x} \bar{z}} + \bar{x}) = x z + x y z + x y z + \bar{x} y = x z + x y z + \bar{x} y = x z + \bar{x} y.$
45. $\overline{\bar{x} + y + x y + \bar{x} + \bar{y}} = \overline{\bar{x} \bar{y} + x y + \bar{x} \bar{y}} = \overline{x \bar{y} + x y + \bar{x} \bar{y}} = \overline{x \bar{y} + x \bar{y} + x y + \bar{x} \bar{y}} = \overline{x \bar{y} + x y + x \bar{y} + \bar{x} \bar{y}} = \overline{x (y + \bar{y}) + (x + \bar{x}) \bar{y}} = \overline{x (y + \bar{y}) + \bar{y} (x + \bar{x})} = \overline{x 1 + \bar{y} 1} = \overline{x + \bar{y}}.$
46. $x = x \cdot 1 = x (x + \bar{x}) = x x + x \bar{x} = x x + 0 = x x.$
47. $x = x + 0 = x + x \bar{x} = (x + x)(x + \bar{x}) = (x + x) \cdot 1 = x + x.$
48. $x + (x y) = x \cdot 1 + x y = x (y + \bar{y}) + x y = x y + x y + x \bar{y} =^* x y + x \bar{y} = x (y + \bar{y}) = x \cdot 1 = x$ [*using idempotent law].
49. $x + 1 = (x + 1) \cdot 1 = (x + 1)(x + \bar{x}) = x + 1 \bar{x} = x + \bar{x} = 1.$
50. True.
51. True.
52. False.
53. True.
54. True.
55. False.
56. True.
57. True.
58. False.
59. True.
60. True.
61. True.
62. The following laws fail: distributive law for addition over multiplication and the two complement laws.
63. Additive property of 0, multiplicative property of complement, distributive law for addition over multiplication, additive property of complement, commutative law for multiplication, multiplicative property of 1.
64. Additive property of complement, multiplicative property of 1, distributive law for addition over multiplication, additive property of complement, multiplicative property of 1.
65. Multiplicative property of 1, additive property of complement, distributive law for multiplication over addition, associative law for addition, additive idempotent law, distributive law for multiplication over addition, additive property of complement, multiplicative property of 1.

66.



67.



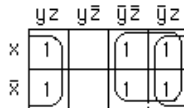
68. The following Karnaugh map yields $x + \bar{z}$.



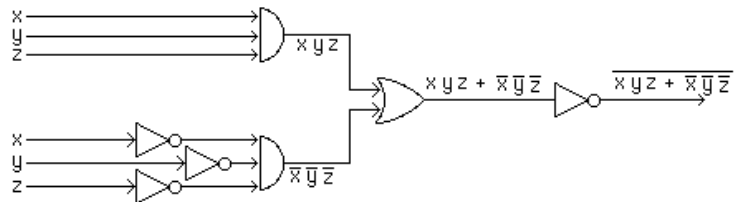
69. The following Karnaugh map yields $x\bar{y} + xz + \bar{y}z$.



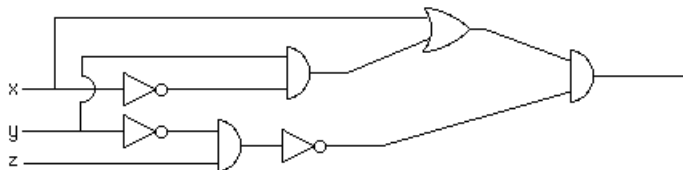
70. The following Karnaugh map yields $\bar{y} + z$.



71. The function $f(x, y, z) = \overline{(xyz + \bar{x}\bar{y}\bar{z})}$, with the following gate diagram, gives the desired output.



72.



73. $(\bar{y}z)(x + \bar{x}y) = (y + \bar{z})(x + \bar{x}z) = xz + x\bar{z} + y\bar{x}y + \bar{x}y\bar{z} = xy + x\bar{z} + \bar{x}y + \bar{x}y\bar{z} = xy + \bar{x}y + x\bar{z} + \bar{x}y\bar{z} = (x + \bar{x})y + x\bar{z} + \bar{x}y\bar{z} = y + \bar{x}y\bar{z} + x\bar{z} = y + x\bar{z}$.

74. The Karnaugh map for F is drawn here, with “d” used for the don't care conditions (i.e., the bit strings representing the numbers 10 through 15). All 1's are covered by one oval in this map, and hence $F(w, x, y, z) = \bar{z}$.

	yz	$y\bar{z}$	$\bar{y}z$	$\bar{y}\bar{z}$
wx	d	d	d	d
$w\bar{x}$	d	d	1	
$\bar{w}\bar{x}$		1	1	
$\bar{w}x$		1	1	

75. The Karnaugh map for G is drawn here, with “d” used for the don’t care conditions (that is, the bit strings representing the numbers 10 through 15). All 1’s are covered by one oval, and hence $G(w, x, y, z) = z$.

	yz	$y\bar{z}$	$\bar{y}z$	$\bar{y}\bar{z}$
wx	d	d	d	d
$w\bar{x}$	d	d		1
$\bar{w}\bar{x}$	1			1
$\bar{w}x$	1			

76.

1	xyz	111	(1,2)	yz	-11	(2,3,4,5)	\bar{x}	0--
2	$\bar{x}yz$	011	(2,3)	xy	01-			
3	$\bar{x}y\bar{z}$	010	(2,4)	$\bar{x}z$	0-1			
4	$\bar{x}\bar{y}z$	001	(3,5)	$\bar{x}\bar{z}$	0-0			
5	$\bar{x}\bar{y}\bar{z}$	000	(4,5)	$\bar{x}\bar{y}$	00-			

The products that were not used to form products in fewer variables are yz and x . This yields

	$\bar{x}yz$	$\bar{x}y\bar{z}$	$\bar{x}\bar{y}z$	$\bar{x}\bar{y}\bar{z}$	xyz
yz	×		×		×
\bar{x}	×	×	×	×	

To cover the original five minterms, we use $\bar{x} + yz$.

77.

1	$wxyz$	1111	(1,2)	wxy	111-	(1,2,3,4)	wy	1-1-
2	$wxy\bar{z}$	1110	(1,3)	wyz	1-11	(1,2,3,4)	wy	1-1-
3	$w\bar{x}yz$	1011	(2,4)	$wy\bar{z}$	1-10	(3,5,6,7)	$\bar{x}z$	-0-1
4	$w\bar{x}y\bar{z}$	1010	(3,4)	$w\bar{x}y$	101-	(3,5,6,7)	$\bar{x}z$	-0-1
5	$\bar{w}\bar{x}yz$	0011	(3,5)	$\bar{x}yz$	-011			
6	$\bar{w}\bar{x}y\bar{z}$	1001	(3,6)	$w\bar{x}z$	10-1			
7	$\bar{w}\bar{x}\bar{y}z$	0001	(5,7)	$\bar{w}\bar{x}z$	00-1			
			(6,7)	$\bar{x}\bar{y}z$	-001			

The products that were not used to form products in fewer variables are wy and $\bar{x}z$. This yields

	$wxyz$	$wxy\bar{z}$	$w\bar{x}yz$	$w\bar{x}y\bar{z}$	$\bar{w}\bar{x}yz$	$\bar{w}\bar{x}y\bar{z}$	$\bar{w}\bar{x}\bar{y}z$
wy	×	×	×	×			
$\bar{x}z$			×		×	×	×

To cover the original seven minterms, use $wy + \bar{x}z$.

Questions for Chapter 13

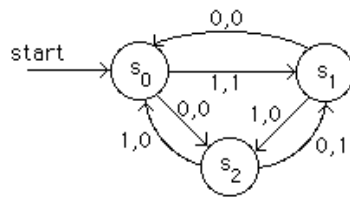
- Suppose $A = \{0, 1\}$. Describe all strings belonging to A^* .

2. Suppose a phrase-structure grammar has productions $S \rightarrow S0, S \rightarrow A1, A \rightarrow 0$. Find a derivation of 01.
3. Suppose a phrase-structure grammar has productions $S \rightarrow S0, S \rightarrow A1, A \rightarrow 0$. Find a derivation of 0100.
4. Suppose a phrase-structure grammar has productions $S \rightarrow S0, S \rightarrow A1, A \rightarrow 0$. Find a derivation of 010.
5. Suppose a phrase-structure grammar has productions $S \rightarrow 1S0, S \rightarrow 0A, A \rightarrow 0$. Find a derivation of 00.
6. Suppose a phrase-structure grammar has productions $S \rightarrow 1S0, S \rightarrow 0A, A \rightarrow 0$. Find a derivation of 1000.
7. Suppose a phrase-structure grammar has productions $S \rightarrow 1S0, S \rightarrow 0A, A \rightarrow 0$. Find a derivation of 110000.
8. Suppose a phrase-structure grammar has productions $S \rightarrow S11, S \rightarrow 0A, S \rightarrow A1, A \rightarrow 0$. Find a derivation of 01.
9. Suppose a phrase-structure grammar has productions $S \rightarrow S11, S \rightarrow 0A, S \rightarrow A1, A \rightarrow 0$. Find a derivation of 0011.
10. Suppose a phrase-structure grammar has productions $S \rightarrow S11, S \rightarrow 0A, S \rightarrow A1, A \rightarrow 0$. Find a derivation of 011111.
11. Find a production of the form " $A \rightarrow \underline{\quad}$ " such that $S \rightarrow 0A, A \rightarrow \underline{\quad}$ produces $\{00\}$.
12. Find a production of the form " $A \rightarrow \underline{\quad}$ " such that $S \rightarrow 1S, S \rightarrow 0A, A \rightarrow \underline{\quad}$ produces $\{1^n 00 \mid n \geq 0\}$.
13. Find a set of two productions that produces $\{1^{2n} \mid n > 0\}$.
14. Let G be the phrase-structure grammar with vocabulary $V = \{A, B, a, b, S\}$, terminal element set $T = \{a, b\}$, start symbol S , and production set $P = \{S \rightarrow ABa, S \rightarrow Ba, A \rightarrow aB, AB \rightarrow b, B \rightarrow ab\}$. Which of these are derivable from ABa ? (1) ba , (2) abb , (3) aba , (4) b , (5) $aababa$.
15. Let G be the phrase-structure grammar with vocabulary $V = \{A, B, a, b, S\}$, terminal element set $T = \{a, b\}$, start symbol S , and production set $P = \{S \rightarrow ABa, S \rightarrow Ba, A \rightarrow aB, AB \rightarrow b, B \rightarrow ab\}$. Which of these are derivable from A ? (1) $babaa$, (2) aab , (3) bba .
16. Let G be the phrase-structure grammar with vocabulary $V = \{A, B, a, b, S\}$, terminal element set $T = \{a, b\}$, start symbol S , and production set $P = \{S \rightarrow ABa, S \rightarrow Ba, A \rightarrow aB, AB \rightarrow b, B \rightarrow ab\}$. Which of these are derivable from S ? (1) ba , (2) ab , (3) $baab$, (4) $aababa$, (5) aba .
17. Let G be the phrase-structure grammar with vocabulary $V = \{A, B, 0, 1, S\}$, terminal elements $T = \{0, 1\}$, start symbol S , productions $P = \{S \rightarrow AB0, AB \rightarrow 1, A \rightarrow 0, B \rightarrow AB\}$. Which of these are derivable from S ? (1) 000, (2) 11, (3) 010, (4) 0000, (5) 0001, (6) 110, (7) 0010.
18. Suppose $V = \{S, A, a, b\}$, $T = \{a, b\}$, and S is the start symbol. Find a set of productions that includes $S \rightarrow Aa$ and $A \rightarrow a$ and generates the language $\{a, aa\}$.
19. Suppose $V = \{S, A, a, b\}$, $T = \{a, b\}$, and S is the start symbol. Find a set of productions that includes $S \rightarrow Aa$ and $A \rightarrow a$ and generates the language $\{a, b, ba, baa\}$.
20. The productions of a phrase-structure grammar are $S \rightarrow S1, S \rightarrow 0A, A \rightarrow 1$. Find a derivation of 0111.
21. What language is generated by the phrase-structure grammar if the productions are $S \rightarrow S11, S \rightarrow \lambda$, where S is the start symbol?
22. What is the language generated by the grammar with productions $S \rightarrow SA, S \rightarrow 0, A \rightarrow 1A$, and $A \rightarrow 1$, where S is the start symbol?
23. Find a grammar for the set $\{0^{2n}1^n \mid n \geq 0\}$.

In questions 24–29 let $V = \{S, A, B, 0, 1\}$ and $T = \{0, 1\}$. For each set of productions determine whether the resulting grammar G is

- | | |
|---------------------------------------|--------------------------------------|
| (i) type 0 grammar, but not type 1, | (ii) type 1 grammar, but not type 2, |
| (iii) type 2 grammar, but not type 3, | (iv) type 3 grammar. |

- 24. $S \rightarrow A10, AB \rightarrow 0$.
- 25. $S \rightarrow B, A \rightarrow B, B \rightarrow A$.
- 26. $S \rightarrow AB, A \rightarrow 0B1, 0B1 \rightarrow 0$.
- 27. $S \rightarrow 1A, A \rightarrow 1, S \rightarrow \lambda$.
- 28. $S \rightarrow 1AB, AB \rightarrow 0B, B \rightarrow 0, A \rightarrow 1B$.
- 29. $S \rightarrow 0B, B \rightarrow 1A, B \rightarrow 0, A \rightarrow 0B$.
- 30. Construct a finite-state machine that models a vending machine accepting only quarters that gives a container of orange juice when 50 cents has been deposited, followed by a button being pushed. (The possible inputs are quarters and the button, and the possible outputs are nothing, orange juice, and a quarter. The machine returns any extra quarters.)
- 31. What is the output produced by this finite-state machine when the input string is 11101?



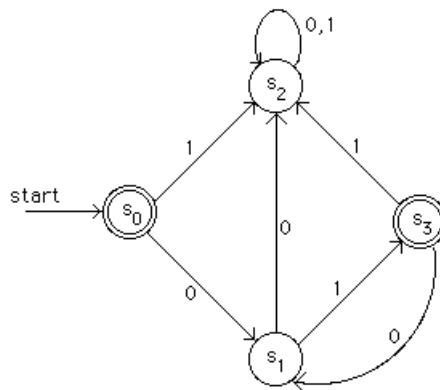
- 32. Construct a finite-state machine with output that produces a 1 if and only if the last 3 input bits read are 0's.
- 33. Suppose that $A = \{1, 11, 01\}$ and $B = \{0, 10\}$. Find AB .
- 34. Suppose that $A = \{1, 11, 01\}$ and $B = \{0, 10\}$. Find BA .

In questions 35–38 determine the output for each input string, using this state table.

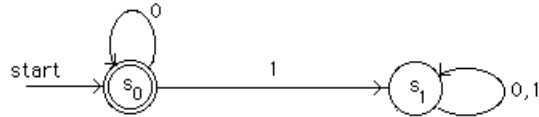
	<i>f</i>		<i>g</i>	
	Input		Input	
	0	1	0	1
s_0	s_1	s_2	1	1
s_1	s_2	s_0	0	0
s_2	s_1	s_3	1	1
s_3	s_3	s_1	0	1

- 35. 1111.
- 36. 10111.
- 37. 000.
- 38. 11000.

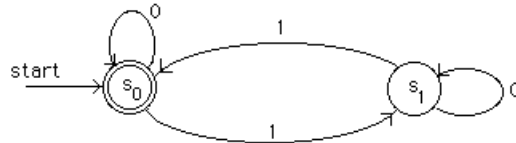
- 39. Let $A = \{1, 10\}$. Which strings belong to A^* ?
- 40. Find the set recognized by this deterministic finite-state machine.



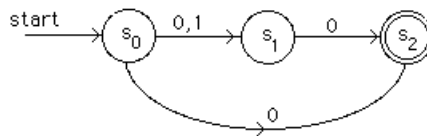
- 41. Find all strings recognized by this deterministic finite-state automaton.



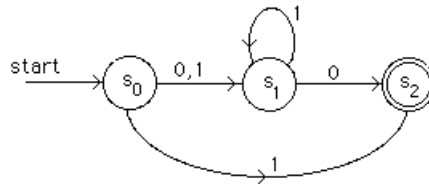
42. Find all strings recognized by this deterministic finite-state automaton.



43. Find the language recognized by this nondeterministic finite-state automaton.



44. Find the language recognized by this nondeterministic finite-state automaton.



45. Let $A = \{0, 11\}$. Find A^2 .

46. Let $A = \{0, 11\}$. Find A^3 .

47. Find the Kleene closure of $A = \{1\}$.

48. Find the Kleene closure of $A = \{00\}$.

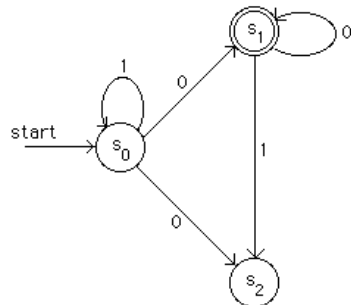
49. Find the Kleene closure of $A = \{0, 1, 2\}$.

50. Which strings belong to the set represented by the regular expression $0^* \cup 11$?

51. Construct a finite-state automaton that recognizes all strings that end with 11.

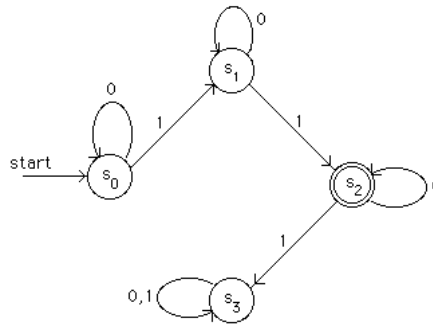
52. Construct a finite-state automaton that recognizes the set represented by the regular expression 10^* .

53. Find a deterministic finite-state automaton equivalent to the following nondeterministic finite-state machine.

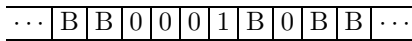


54. Which strings belong to the regular set represented by the regular expression (1^*01^*0) ?

- 55. Determine if 1101 belongs to the regular set $\mathbf{1^*0^*1}$.
- 56. Determine if 1101 belongs to the regular set $\mathbf{(0\cup 1)^*1}$.
- 57. Determine if 1101 belongs to the regular set $\mathbf{(11)^*0^*(11)^*}$.
- 58. Determine if 1101 belongs to the regular set $\mathbf{1(10)^*1^*}$.
- 59. Determine if 1101 belongs to the regular set $\mathbf{(01)^*(11)^*(01)^*}$.
- 60. Determine if 1101 belongs to the regular set $\mathbf{11(00)^*(10)^*}$.
- 61. Determine if 1101 belongs to the regular set $\mathbf{(111)^*(01)^*}$.
- 62. Which strings are recognized by the following finite-state automaton?

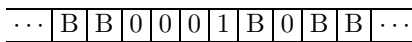


- 63. For the following Turing machines T , find the final tape when T is run on the following tape, beginning in the initial position (the first nonzero entry from the left):



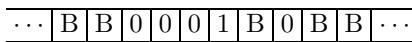
$(s_0, 0, s_0, 0, R), (s_0, 1, s_1, 0, R), (s_1, 0, s_1, 1, R), (s_1, 1, s_2, 1, L), (s_1, B, s_1, 1, L)$.

- 64. For the following Turing machine T , find the final tape when T is run on the following tape, beginning in the initial position (the first nonzero entry from the left):



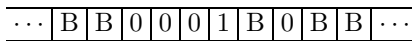
$(s_0, 0, s_0, 1, R), (s_0, 1, s_1, 0, R), (s_1, 1, s_2, 1, R), (s_1, B, s_0, 0, R)$.

- 65. For the following Turing machine T , find the final tape when T is run on the following tape, beginning in the initial position (the first nonzero entry from the left):



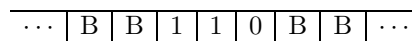
$(s_0, 0, s_1, 1, R), (s_0, 1, s_1, 1, L), (s_1, 0, s_0, 1, L)$.

- 66. For the following Turing machine T , find the final tape when T is run on the following tape, beginning in the initial position (the first nonzero entry from the left):



$(s_0, 0, s_2, 0, R), (s_0, B, s_0, 1, R), (s_1, 0, s_2, 1, R), (s_2, 0, s_1, 1, L), (s_2, 1, s_0, 1, R)$.

- 67. Consider the Turing machine $T: (s_0, 0, s_1, 1, R), (s_0, 1, s_1, 1, R), (s_1, 0, s_0, 1, L), (s_1, 1, s_0, 0, R), (s_0, B, s_1, 1, R)$. For the following tape, determine the final tape when T halts, assuming that T begins in state s_0 at the leftmost nonblank symbol.



- 68. Consider the Turing machine $T: (s_0, 0, s_1, 1, R), (s_0, 1, s_1, 1, R), (s_1, 0, s_0, 1, L), (s_1, 1, s_0, 0, R), (s_0, B, s_1, 1, R)$. For the following tape, determine the final tape when T halts, assuming that T begins in state s_0 at the leftmost

nonblank symbol.

...	B	B	0	0	0	B	B	...
-----	---	---	---	---	---	---	---	-----

69. Consider the Turing machine $T: (s_0, 0, s_1, 1, R), (s_0, 1, s_1, 1, R), (s_1, 0, s_0, 1, L), (s_1, 1, s_0, 0, R), (s_0, B, s_1, 1, R)$. For the following tape, determine the final tape when T halts, assuming that T begins in state s_0 at the leftmost nonblank symbol.

...	B	B	1	0	1	B	B	...
-----	---	---	---	---	---	---	---	-----

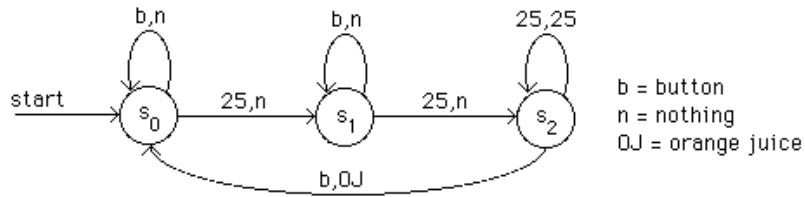
70. Construct a Turing machine that computes $f(n) = n + 2$, where $n \geq 0$.
71. Construct a Turing machine that computes $f(n_1, n_2) = n_2 + 1$, where $n_1, n_2 \geq 0$.

Answers for Chapter 13

1. A^* consists of all strings of 0's and 1's, including the empty string.
2. $S \Rightarrow A1 \Rightarrow 01$.
3. $S \Rightarrow S0 \Rightarrow S00 \Rightarrow A100 \Rightarrow 0100$.
4. $S \Rightarrow S0 \Rightarrow A10 \Rightarrow 010$.
5. $S \Rightarrow 0A \Rightarrow 00$.
6. $S \Rightarrow 1S0 \Rightarrow 10A0 \Rightarrow 1000$.
7. $S \Rightarrow 1S0 \Rightarrow 11S00 \Rightarrow 110A00 \Rightarrow 110000$.
8. $S \Rightarrow A1 \Rightarrow 01$.
9. $S \Rightarrow S11 \Rightarrow 0A11 \Rightarrow 0011$.
10. $S \Rightarrow S11 \Rightarrow S1111 \Rightarrow A11111 \Rightarrow 011111$.
11. $A \rightarrow 0$.
12. $A \rightarrow 0$.
13. $S \rightarrow S11, S \rightarrow 11$.
14. (1), (5).
15. (2).
16. (1), (4), (5).
17. (3), (7).
18. $S \rightarrow Aa, A \rightarrow a, S \rightarrow a$.
19. $S \rightarrow Aa, A \rightarrow a, S \rightarrow aA, A \rightarrow ab$.
20. Apply the production $S \rightarrow S1$ twice to obtain $S11$. Then apply $S \rightarrow 0A$ to obtain $0A11$. Then apply $A \rightarrow 1$ to obtain 0111 .
21. The language generated is the set of all strings consisting of an even number of 1's and no other symbols.
22. The set of all bit strings that consist of a 0 followed by an arbitrary number of 1's.
23. Use the grammar with productions $S \rightarrow 00S1$ and $S \rightarrow \lambda$, where S is the start symbol.
24. *i*.
25. *iii*.
26. *i*.
27. *iv*.
28. *ii*.

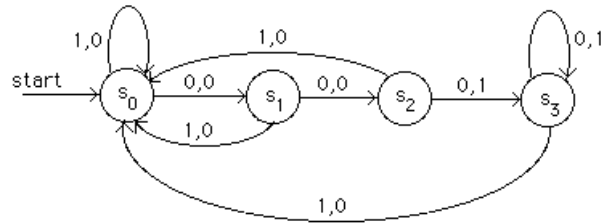
29. *iv.*

30.



31. 10000.

32.



33. $AB = \{10, 110, 1110, 010, 0110\}$.

34. $BA = \{01, 011, 001, 101, 1011, 1001\}$.

35. 1110.

36. 11011.

37. 101.

38. 11000.

39. The strings in A^* are those in which each 0 is preceded by at least one 1.

40. The set represented by $(01)^*$.

41. All bit strings with no 1's.

42. All bit strings with an even number of 1's.

43. $\{0, 00, 10\}$.

44. $\{1, 01^n 0, 1^n 0 \mid n \geq 0\}$.

45. $\{00, 011, 110, 1111\}$.

46. $\{000, 0011, 01111, 0110, 1100, 11011, 11110, 111111\}$.

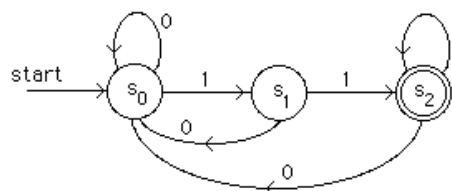
47. $\{1^n \mid n = 0, 1, 2, \dots\}$.

48. $\{0^{2n} \mid n = 0, 1, 2, \dots\}$.

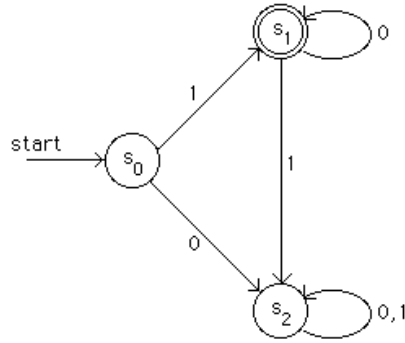
49. All strings of 0's, 1's, and 2's.

50. The bit strings consisting of all 0's (including the empty string) and the string 11.

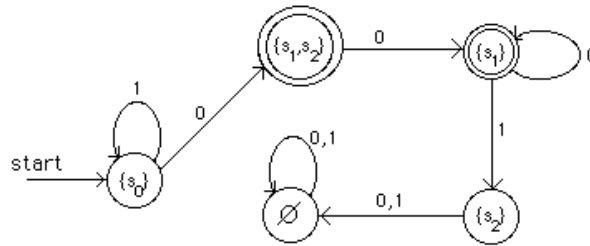
51.



52.



53.



54. The strings containing an even number of 0's and not ending with a 1.

55. Yes.

56. Yes.

57. No.

58. Yes.

59. Yes.

60. No.

61. No.

62. Strings containing exactly two 1's.

63.

...	B	B	0	0	0	1	1	0	B	B	...
-----	---	---	---	---	---	---	---	---	---	---	-----

64.

...	B	B	1	1	1	0	0	1	B	B	...
-----	---	---	---	---	---	---	---	---	---	---	-----

65.

...	B	B	1	1	0	1	B	0	B	B	...
-----	---	---	---	---	---	---	---	---	---	---	-----

66.

...	B	B	1	1	0	1	1	0	B	B	...
-----	---	---	---	---	---	---	---	---	---	---	-----

67.

...	B	B	1	0	1	B	B	...
-----	---	---	---	---	---	---	---	-----

68.

...	B	B	1	1	1	B	B	...
-----	---	---	---	---	---	---	---	-----

69.

...	B	B	1	0	1	B	B	...
-----	---	---	---	---	---	---	---	-----

70. $(s_0, 1, s_1, 1, R), (s_1, 1, s_1, 1, R), (s_1, B, s_2, 1, R), (s_2, B, s_3, 1, R).$

71. $(s_0, 1, s_1, B, R), (s_1, 1, s_1, B, R), (s_1, *, s_2, 1, R).$